Relay Solutions

1. (a) Consider the smallest 4-digit number that is the sum of three consecutive natural numbers. What is the smallest of these three consecutive natural numbers?

   Solution: 333
   The four digit number satisfies $x+(x+1)+(x+2)=3(x+1)$, for some integer x. So we are looking for the smallest four digit number that is divisible by 3, which is 1002. Then $x+1=\frac{1002}{3}$, and so $x=333$.

   (b) Half of a number x is Z less than twice the square of the sum of the first five prime numbers, where $Z=\frac{a}{9}$. What is x?

   Solution: 3062
   We have $\frac{x}{2}=2(2+3+5+7+11)^2-Z$. So $x=2(2(784)-Z)=3136-2Z=3136-2(37)=3062$, since $Z=\frac{333}{9}=37$.

   (c) If the average of five consecutive integers is Z, where $Z=\frac{b-32}{30}-100$, what is the largest of these five consecutive integers?

   Solution: 3
   We want the largest of five consecutive integers whose average is Z. Let n denote the smallest of these five integers.
   Then we have $\frac{n+(n+1)+(n+2)+(n+3)+(n+4)}{5}=\frac{5n+10}{5}=\frac{5(n+2)}{5}=n+2$. If this average is Z, then $n=Z-2$.

   So the largest of the five numbers is $n+4=Z-2+4=Z+2$.

   We have $Z=\frac{3062-32}{30}-100=\frac{3030}{30}-100=101-100=1$. So the five consecutive integers are $-1,0,1,2,3$, the largest of which is 3.

   (d) Let $Z=3c$. A student takes a 50 question test, where Z points are awarded for each correct answer and 6 points are deducted for each wrong answer. If the student answers every question and scores a 30, how many questions did that student answer correctly?

   Solution: 22
   Let x denote the number of questions that the student answered correctly. Then the student's score is $Zx-6(50-x)=Zx-300+6x=30$.
   So $x=\frac{330}{Z+6}$. We have $Z=3(3)=9$. Thus $x=\frac{330}{9+6}=\frac{330}{15}=22$.

2. (a) You roll two dice, one colored red and one colored gold. What is the probability you roll a sum of 8.

Solution: $\frac{5}{36}$

The sample space is $S = \{(1,1),(1,2),\dots(1,6),(2,1),(2,2),\dots(2,6),\dots(6,6)\}$ and $|S| = 6 \times 6 = 36$.
We can sum to 8 as: $2+6, 3+5, 4+4$. Since the dice are colored we can have red 2, gold 6 or red 6 gold 2. Likewise we can have a red 3, gold 5 or a red 5, gold 3. There is only one possibility for $4+4$, as red 4, gold 4. For a total of 5 possibilities.
So we have, $P = \frac{5}{36}$

(b) Let $Z = 36a$.

On an exam, Bob is told to subtract Z from a given number and then divide by 9. Instead he subtracted 9 and divided by Z to get 46. If the problem was worked correctly, what would the answer be?

Solution: 26

Let x be the given number. Then

$$\frac{x-9}{Z} = 46 \Rightarrow x - 9 = 46Z \Rightarrow x = 46Z + 9$$

The correct answer is then

$$\frac{46Z + 9 - Z}{9} = \frac{45Z + 9}{9} = 5Z + 1$$

We are given $Z = 36a = 36 \cdot \frac{5}{36} = 5$.
Giving an answer of $5(5) = 25 + 1 = 26$

(c) Let $Z = b - 17$.

Solve the following system of equations for c.

$$2x - c = Z$$
$$3x + 4c = -14.$$

Solution: -5

Multiplying the top equation by 4 we have

$$8x - 4c = 4Z$$
$$3x + 4c = -14$$

Adding the equations we have $11x = 4Z - 14$. So

$$x = \frac{4Z - 14}{11}.$$

Solving the top equation for c, we have

$$c = 2x - Z = 2\left(\frac{4Z - 14}{11}\right) - Z = \frac{8Z - 28 - 11Z}{11} = \frac{-3Z - 28}{11}.$$

Now $Z = 26 - 17 = 9$, so

$$c = \frac{-3(9) - 28}{11} = -5$$

(d) Let $Z = c + 8$

Suppose a set $A = \{1, 2, 3, \ldots, 12\}$. How many subsets of size Z are there?

Solution: 220

If $Z = 1$, then there are 12 possible subsets, $\{1\}, \{2\}, \ldots \{11\}, \{12\}$.

If $Z = 2$, then there are 66 possible subsets. Choose any element, which we can do in 12 ways. There are 11 elements left to choose from. Which gives us 12×11 ways. But we have now counted both $\{a, b\}$ and $\{b, a\}$ as different subsets, i.e. we've over counted. So there are $\frac{12 \times 11}{2}$ possible subsets. In fact this is $C(12, 2)$.

This pattern will continue and we can show that the number of subsets of size Z is $C(12, Z)$.
We are given $Z = -5 + 8 = 3$.
The number of subsets is

$$C(12, 3) = \frac{12!}{9!3!} = \frac{12 \times 11 \times 10}{3 \times 2} = 220$$

3. (a) What is the largest 3-digit integer whose square root is prime?

Solution: 961
We know that $30^2 = 900$. Checking further, we have $31^2 = 961$, and $32^2 = 1024$, which is too large. Thus 961 is the largest 3-digit integer whose square root is prime.

(b) Let $Z = \sqrt{a} - 10$. A rectangular box has length Z cm, width 12 cm, and height 16 cm. Find the length of the longest line segment that can be contained in this box.

Solution: 29
The longest line segment will be along a diagonal from opposite corners of the box; for example from the front lower left corner to the top upper back corner. Let x denote the length of this diagonal. Using the Pythagorean theorem to find the diagonal length along one side, we have $12^2 + 16^2 = 400$.
Then $Z^2 + 400 = x^2$, so $x = \sqrt{Z^2 + 400}$. We have $Z = 31 - 10 = 21$. Thus, $x = \sqrt{21^2 + 400} = \sqrt{441 + 400} = \sqrt{841}$. So the longest line segment has length 29.

(c) Let $Z = b - 23$. Starting with a number x, form a second number that is Z more than x, a third number that is $2Z$ more than x, and continuing this pattern until we have a sixth number which is $5Z$ more than x. If the sum of these six numbers is the least common multiple of 117 and 72, what is x?

Solution: 141
Note that $117 = 9 \cdot 13$ and $72 = 8 \cdot 9$, so the LCM of these numbers is $8 \cdot 9 \cdot 13 = 936$. Thus, we want $x+(x+Z)+(x+2Z)+(x+3Z)+(x+4Z)+(x+5Z) = 6x+15Z = 936$. Then $x = \dfrac{936 - 15Z}{6}$.
We see that $Z = 29 - 23 = 6$. So $x = \dfrac{936 - 15(6)}{6} = \dfrac{846}{6} = 141$.

(d) Let $Z = c + 87$. A book was printed that uses a total of Z digits for the page numbers, where $Z > 190$. If the first page is numbered 1, how many numbered pages are in the book?

Solution: 112
The first nine pages (1-9) use nine digits. The next ninety pages (10-99) use two digits per page for a total of 180 digits. Thus, the first ninety nine pages use 189 digits. Since $Z > 190$, let t denote the number of pages with three-digit pages numbers. Then $189 + 3t = Z$. So $t = \dfrac{Z - 189}{3}$.
Then, letting x denote the total number of pages, we have $x = 99 + t = 99 + \dfrac{Z - 189}{3}$.
Note that $Z = 141 + 87 = 228$. Then the number of pages is $x = 99 + \dfrac{228 - 189}{3} = 99 + 13 = 112$.

4. (a) Consider the following true statements:
   - Each heist is made by three minions.
   - Each minion makes exactly two heists.
   - Every pair of minions make at most one common heist.
   - There is at least one heist.

   What is the minimal number of possible heists?

   Solution: 4

   By Axiom 4, there is at least one heist, h_1. Axiom 1 says that h_1 must be made by three minions m_1, m_2, m_3. By Axiom 2 m_1 must make an additional heist h_2. This heist must be made by two additional minions. They cannot be m_2 or m_3 or Axiom 3 would be violated, since they would have made two common heists with m_1. So h_2 is made by m_4 and m_5. Now m_2 must also make a second heist by Axiom 2, say h_3. Likewise m_3 must make another heist. This heist cannot be h_3 or m_2 and m_3 would make two common heists, violating Axiom 3. So m_3 must make heist h_4. Thus we have shown there are at least four heists, $h_1, \ldots h_4$. $\square$

   (b) Let $Z = a + 12$.

   Judy Hopps has been put on parking duty in Zootopia. Judy is determined to give more than 200 tickets, in fact she gives 215 tickets. There are Z officers on parking duty and an average of 80 tickets have been given. If we remove Judy, what is the average number of tickets for all other officers on parking duty?

   Solution: 71

   The number of police officers is Z. Let A_t be the total average and A_r be the average after removing Judy's number of tickets.
   Then

$$A_t = \frac{A_r(Z-1) + 215}{Z} = 80.$$

   So

$$A_r(Z-1) + 215 = 80Z$$

$$A_r = \frac{80Z - 215}{Z-1}.$$

   We have

$$Z = 4 + 12 = 16.$$

   So

$$A_r = \frac{80(16) - 215}{15} = \frac{1065}{15} = 71$$

   (c) Let $Z = \dfrac{b+1}{12}$

Hiro needs to reprogram Baymax. In Baymax's chip, a word is translated into a sequence of 0's and 1's. How many sequences of length Z are there if there must be at most two 0's.

Solution: 22

We can consider which positions in the sequence are 0's. To do this we just select the positions using combinations.

The number of ways there can be no 0's is $\binom{Z}{0} = 1$.

The number of ways there can be one 0 is $\binom{Z}{1} = Z$.

The number of ways there can be exactly two 0's is $\binom{Z}{2} = \frac{Z(Z-1)}{2}$.

So in total we have

$$1 + Z + \frac{Z(Z-1)}{2}$$

possible sequences.

Since we are given $Z = \frac{71+1}{12} = 6$.

Thus we have

$$1 + 6 + \frac{6(5)}{2} = 1 + 6 + 15 = 22.$$

(d) Let $Z = c - 17$.

Anna buys a bunch of carrots for Sven and the other reindeers. Sven eats five times as many as one reindeer. The remaining Z reindeer eat 3 each. What is the smallest number of carrots possible in the bunch that Anna bought?

Solution: 21

There are $Z + 2$ reindeer. Each of the Z reindeer eat three carrots. Now Sven eats 5 times as many as reindeer y. So in total there are $3Z + y + 5y = 3Z + 6y$ carrots.

We are given $Z = 22 - 17 = 5$.

The smallest number of carrots is when $y = 1$. So there are $3(5) + 6 = 21$ carrots.

5. (a) In how many distinct ways can all of the letters in $BUMBLEBEE$ be arranged so that M and L are adjacent?

Solution: 2240
There are nine letters, of which 3 are B and 3 are E. So there are $\frac{9!}{3!3!}$ distinct ways to arrange all of the letters. If we must have ML or LM, then there are $2\frac{8!}{3!3!} = 2 \cdot 8 \cdot 7 \cdot 5 \cdot 4 = 2240$ ways.

(b) Let $Z = a - 2236.5$. Suppose that John leaves his house at 7:00 and walks 2.5 miles to school at a rate of 3 miles per hour. If his sister Kayla leaves 5 minutes after he did and walks the same route as John at Z miles per hour, how many minutes will it take Kayla to catch up with John?

Solution: 35
We have distance = (rate)(time). So John's distance is $3t$, where t is the number of hours since John started walking. Kayla's distance is $Z(t-\frac{5}{60}) = Z(t-\frac{1}{12})$. Since the siblings will meet when they have walked the same distance, we want $3t = Z(t-\frac{1}{12})$, or $3t = Zt - \frac{Z}{12}$, which gives $\frac{Z}{12} = (Z-3)t$. Hence $t = \frac{Z}{12(Z-3)}$.
Since $Z = 2240 - 2236.5 = 3.5$, $t = \frac{3.5}{12(3.5-3)} = \frac{3.5}{6}$ hours, which is $\frac{3.5}{6}(60) = 35$ minutes.

(c) Let $Z = \frac{b}{7}$. It's 1 AM and you can't sleep, so you decide to calculate the amount of money that you would earn in the next 24 hours if you made \$$Z$ every time that the hands of a non-digital clock were 90^o apart. How much money would you make (in dollars)?

Solution: 220
Right angles happen twice per hour with two times that we have to be careful to not count twice: 3 o'clock and 9 o'clock. This gives 22 right angles in 12 hours, and so we have 44 right angles in 24 hours. Thus the amount of money made is $44Z$. Now $Z = \frac{35}{7} = 5$. Thus, you would make $44(Z) = 44(5) = 220$ dollars.

(d) Let $Z = \frac{c}{10} - 13$. If there are Z students who want to begin a club, in how many ways can four different people be chosen for president, vice president, secretary, and a treasurer if three of the members refuse to serve in these offices?

Solution: 9
We have $Z - 3$ people to choose from for four positions, so it's $(Z-3)(Z-4)(Z-5)(Z-6) = 6(5)(4)(3) = 360$, since $Z = 22 - 13 = 9$.

6. (a) Barney, Robin, Ted, Lily and Marshall are racing through the streets of NYC to get to a restaurant where Woody Allen has been spotted. In how many possible orders can they arrive at the restaurant?

> **Solution: 120** Any of the five characters can arrive first. There are four options for who arrives second, since you cannot arrive in both first and second. Continuing in this manner we see there are
>
> $$5 \times 4 \times 3 \times 2 \times 1 = 5! = 120.$$

(b) Let $Z = \dfrac{a}{30}$

Marshall is waiting for his results on the Bar Exam. His password to receive his results is a string of letters followed by a string of the integers 0-9. If there are no restrictions on the numbers, how many ways can he create a string of Z numbers?

> **Solution: 10,000**
>
> Note: there are 10 numbers 0-9.
> Since there are no restrictions there are
>
> $$\underbrace{10 \times 10 \times 10 \cdots \times 10}_{\text{Z times}} = 10^Z$$
>
> We are given
>
> $$Z = \frac{120}{30} = 4.$$
>
> So there are
>
> $$10^4 = 10,000$$
>
> ways to arrange the numbers.

(c) Let $Z = \sqrt{b}$

Lily is working as an art consultant for the Captain. She finds an elephant painting that she recommends for him to buy. He buys the painting for $\$100,000$. After Z years, the painting is worth $\$4,000,000$. If the interest on the painting used simple annual interest, what was the interest rate (as a percent)?

> **Solution: 40**
>
> Recall simple annual interest is
>
> $$I = Prt.$$
>
> So we have
>
> $$4,000,000 = 100,000rZ.$$

So

$$r = \frac{40}{Z}$$

Now,

$$Z = \sqrt{10,000} = 100.$$

So the interest is

$$r = \frac{40}{100} = .4 = 40\%.$$

(d) Let $Z = c + 37$.

Ted has always wanted to build part of the NYC skyline. He finally has the chance to build the new GNB building, and wonders how tall it should be. He selects 5 other building to compare his to. He knows that the mode of those 5 buildings is 60, the median is 15 more than the mode, and average is Z. He also knows the tallest is the One World Trade Center at 104-stories. How many stories is the 5^{th} building?

Solution: 86

We know that the mode is 60- so there must be at least two 60's. The median is $60 + 15 = 75$. And the maximum is 104. So we have $60, 60, 75, 104$ and the remaining building, say x. Since the average is Z, we have

$$\frac{60 + 60 + 75 + 104 + x}{5} = Z$$

$$x = 5Z - 299.$$

We are given $Z = 40 + 37 = 77$.
So

$$x = 5(77) - 299 = 86.$$

7. (a) Suppose that there are three houses in a row and two children in each house: in one house lives two girls, in another lives two boys, and in the third is a boy and a girl. If you randomly walk into one of these houses and see a boy, what is the probability that the other child in the house is also a boy?

> **Solution:** $\frac{2}{3}$
>
> There are 3 boys and two of them live in the same house. Thus, if you see a boy, there is a $\frac{2}{3}$ chance that you are seeing one of the boys that lives in a house with another boy.

(b) Let $Z = 18a$. Alex has a pair of dice, but instead of the usual labeling, for each die each side has exactly one of the numbers: 2, 3, 5, 7, 12, and 15. If Alex rolls this pair of dice, what is the probability of getting two numbers whose sum is less than Z?

> **Solution:** $\frac{13}{36}$
>
> Since $Z = 12$, we need to count how many of the 36 total sums is less than 12. There are 13 of these: $2+2=4$, $3+2=5$, $5+2=7$, $7+2=9$, $2+3=5$, $3+3=6$, $5+3=8$, $7+3=10$, $2+5=7$, $3+5=8$, $5+5=10$, $2+7=9$, and $3+7=10$. So the probability is $\frac{13}{36}$.

(c) Let $Z = 10(18b - \frac{3}{2})$. Amber's age is currently twice what Joe's age was five years ago. In three years the sum of Joe's and Amber's ages will be Z. How old is Amber today?

> **Solution: 26**
>
> Let x denote half of Amber's current age. Then Joe's current age is $x+5$. In three years we will have $(2x+3)+(x+5+3) = Z$, or $3x+11 = Z$, so $x = \frac{Z-11}{3}$, and Amber's current age is $2(\frac{Z-11}{3})$. Since $Z = 10(18(\frac{13}{36}) - \frac{3}{2}) = 10(\frac{13}{2} - \frac{3}{2}) = 10(5) = 50$, Amber's age is $2(\frac{50-11}{3}) = 2(13) = 26$.

(d) Let $Z = \frac{c}{2} - 10$. A survey was conducted to determine what ice cream flavor customers like, given the choice of vanilla or chocolate. Of those surveyed, $\frac{1}{Z}$ didn't like chocolate, $\frac{1}{5}$ did not like either flavor, $\frac{2}{7}$ didn't like vanilla, and 183 liked both chocolate and vanilla. How many people were surveyed?

> **Solution: 315**
> The proportion of those who didn't like chocolate or vanilla is $\frac{1}{Z} - \frac{1}{5} + \frac{2}{7}$.

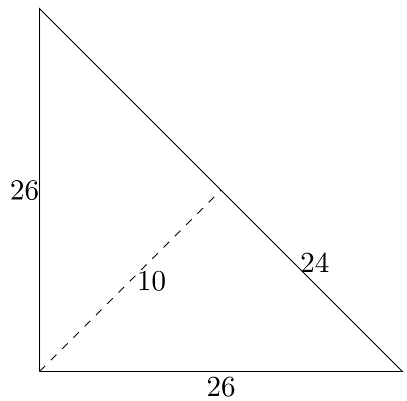
Adding that to the proportion of those who like both chocolate and vanilla, we get $\frac{1}{Z} - \frac{1}{5} + \frac{2}{7} + \frac{183}{x} = 1$, where x is the total number of people surveyed.

Thus $x = \frac{183}{1 + 1/5 - 2/7 - 1/Z}$.

Now $Z = \frac{26}{2} - 10 = 3$, so we have $\frac{1}{3} - \frac{1}{5} + \frac{2}{7} + \frac{183}{x} = \frac{44}{105} + \frac{183}{x} = 1$. Then $\frac{183}{x} = \frac{61}{105}$, which means $x = (183)(105)/61 = 3(105) = 315$.

8. (a) Consider an isosceles triangle with side lengths 26 and base length of 48. Find the area of the triangle.

Solution: 240



Drawing in the height, we have a 10-24-26 right triangle. Then

$$A_1 = \frac{1}{2}bh = \frac{1}{2}(24)(10) = 120.$$

The area of the entire triangle is then $A = 120(2) = 240$.

(b) Let $Z = \dfrac{a}{60}$.

You are making a part iced tea and part lemonade drink. You take two glasses. In the first glass, you pour Z ounces of iced tea. In the second glass you pour Z ounces of lemonade. You pour 1/2 of the first glass into the second. You mix the second glass thoroughly and pour 1/2 back into the first glass. What fraction of the liquid in the first glass is lemonade?

Solution: $\dfrac{2}{5}$

We start out with Z ounces of iced tea in glass 1 and Z ounces of lemonade in glass 2.
After we pour 1/2 of the first glass into the second: we have $Z/2$ ounces of iced tea in glass 1 and $Z/2$ ounces of iced tea plus Z ounces of lemonade in glass 2.
In the next step we pour $(Z/2)/2 = Z/4$ ounces of iced tea and $Z/2$ ounces of lemonade into glass 1.
So in glass one there is now $Z/4 + Z/2 = \dfrac{3Z}{4}$ ounce of iced tea and $Z/2$ ounces of lemonade.
We have $Z = \dfrac{120}{30} = 4$. So we have $\dfrac{3(4)}{4} = 3$ ounces of iced tea and 2 ounces of lemonade.
So fraction of lemonade is 2/5

(c) Let $Z = 5b + 6$.

Compute the sum of all of the roots of $(3x+4)(x-2)+(3x+4)(x-Z)=0$.

Solution: $\frac{11}{3}$
We can factor to get

$$(3x+4)[(x-2)+(x-Z)]=(3x+4)(2x-2-Z)=0.$$

So

$$x=-\frac{4}{3}, x=\frac{2+Z}{2}.$$

We are given $Z=5(2/5)+6=2+6=8$.
Thus the sum is

$$-\frac{4}{3}+\frac{2+8}{2}=-\frac{4}{3}+5=\frac{11}{3}.$$

(d) Let $Z=6c$.

Let Z be the sum of two numbers. If we add 4 to each number and then double the result of each number, then what is the sum of the final two numbers?

Solution: 60

Let x and y be the numbers. Then we have $x+y=Z$. Then $2(x+4)+2(y+4)=2(x+y)+16=2Z+16$.
Since $Z=6\left(\frac{11}{3}\right)=22$, our answer is $2Z+16=2(22)+16=44+16=60$.