1^{st} Annual King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 5 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 1, 11, 16, 18, 34

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. ($\star$) Teams A and B are playing a series of games. Each team has a 50/50 chance of winning any game. If Team A must win two games or Team B must win three games to win the series, find the probability that team A wins the series.

   A. $\frac{11}{16}$ B. $\frac{5}{7}$ C. $\frac{8}{11}$ D. $\frac{3}{5}$ E. $\frac{13}{19}$

2. If $\frac{4^x}{2^{x+y}} = 8$ and $\frac{9^{x+y}}{3^{5y}} = 243$, find the value of xy.

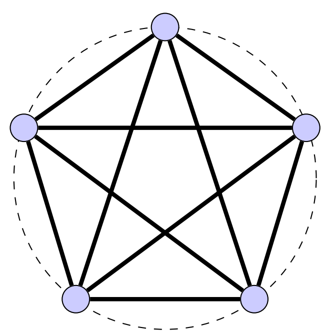
   A. $\frac{12}{5}$ B. 4 C. 6 D. 12 E. -4

3. If $ab \neq 0$ and $|a| \neq |b|$, find the number of distinct values of x satisfying the equation

$$\frac{x-a}{b} + \frac{x-b}{a} = \frac{b}{x-a} + \frac{a}{x-b}$$

   A. zero B. one C. two D. three E. four

4. Suppose five dots are arranged in a circle. If we draw line segments joining each dot to each other dot, then a total of 10 line segments are drawn, as shown in the picture to the right. If 100 dots are arranged in a circle, and line segments are drawn joining each dot to each other dot, how many lines segments are drawn?



   A. 4851 B. 4901 C. 4950 D. 5001 E. 5050

5. At his usual rate a man rows 15 miles downstream in five hours less time than it takes him to return. If he doubles his usual rate, the time downstream is only one hour less than the time upstream. In miles per hour, find the rate of the stream's current.

   A. 2 B. $\frac{5}{2}$ C. 3 D. $\frac{7}{2}$ E. 4

6. The sum of 31 consecutive integers is 2015. What is the largest of these integers?

   A. 70 B. 72 C. 75 D. 80 E. 85

7. Solve the equation $S = \pi rh + \pi r^2$ for r.

   A. $r = \dfrac{-\pi h \pm \sqrt{\pi^2 h^2 - 4\pi S}}{2\pi}$ B. $r = \dfrac{-\pi \pm \sqrt{\pi^2 + 4\pi S}}{2\pi}$ C. $r = \dfrac{-\pi h \pm \sqrt{\pi h + 4\pi S}}{2\pi}$

   D. $r = \dfrac{-\pi h \pm \sqrt{\pi^2 h^2 + 4\pi S}}{2\pi}$ E. None of these

8. Let $y = 2\cos(2x) + 4\cos(x) - 3$. Find the minimum value of y.

   A. -9 B. -7 C. -6 D. -5 E. 3

9. Consider the parabola $y = 2x^2 + 4x + 7$. How many units do you have to shift the parabola horizontally and vertically to obtain the parabola $y = 2x^2 - 6x - 3$?

   A. $\frac{5}{2}, -\frac{25}{2}$ B. $\frac{3}{2}, -\frac{15}{2}$ C. $\frac{1}{2}, -\frac{5}{2}$ D. $-\frac{1}{2}, \frac{5}{2}$ E. $-\frac{5}{2}, \frac{25}{2}$

10. Which of the following expressions is equal to $\dfrac{1}{\sqrt{2}+\sqrt{3}}$?

    A. $\sqrt{2}-\sqrt{3}$ B. $\sqrt{3}-\sqrt{2}$ C. $\sqrt{3}+\sqrt{2}$ D. 1 E. $\sqrt{6}$

11. ($\star$) Assume that a, b, and c are real numbers. If the system

$$\begin{cases} ax + 4y = c \\ x - 2y = 1 \end{cases}$$

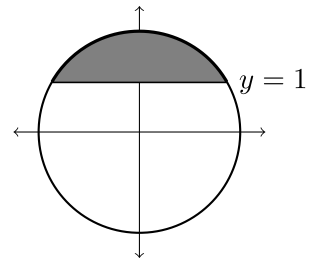
has a solution, and the system

$$\begin{cases} -2x + by = d \\ x - 3y = 2 \end{cases}$$

does not have a solution, which of the following is correct?

A. $a \neq -2$, $b = 6$, $d \neq -4$, c arbitrary B. $a = -2$, $b = 6$, $c \neq -2$, $d = 4$
C. $a = -2$, $c = -2$, b, d arbitrary D. $a = -2$, $b = 6$, c, d arbitrary
E. $a \neq -2$, $b = 6$, c, d arbitrary

12. Consider a circle of radius 2 centered at the origin. Draw a horizontal slice at $y = 1$. What is the area of the gray "cap", as seen to the right?

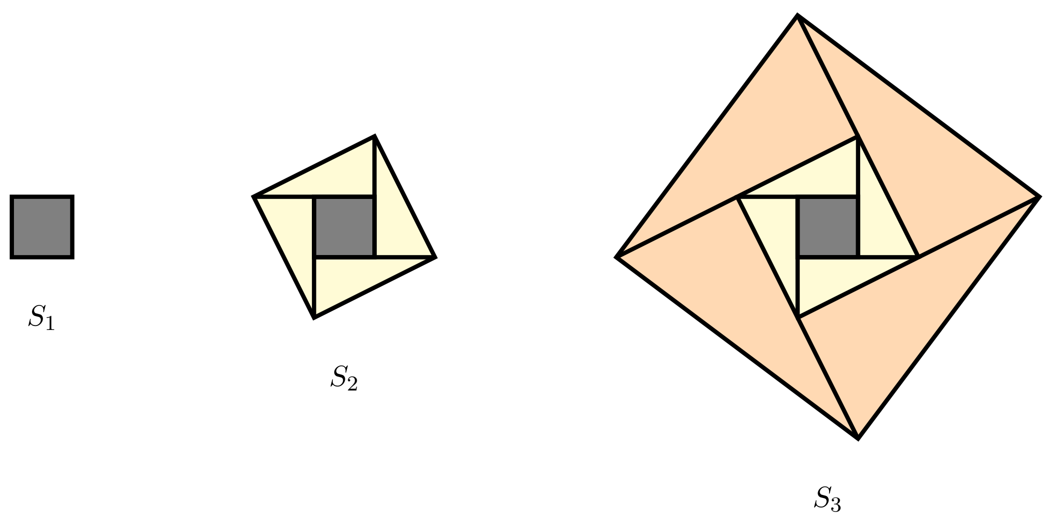


A. $\frac{4\pi}{3} - \sqrt{3}$ B. $\frac{2\pi}{3} - \sqrt{3}$ C. $\frac{4\pi}{3} + \sqrt{3}$ D. $\frac{4\pi}{3}$ E. $\frac{2\pi}{3}$

13. Let $\cos\theta = -\frac{2}{5}$. Find $\tan\theta$ for θ in the 3rd quadrant.

A. $-\frac{\sqrt{21}}{2}$ B. $-\frac{2\sqrt{21}}{21}$ C. $\frac{2\sqrt{21}}{21}$ D. $\frac{\sqrt{21}}{2}$ E. $-\frac{5\sqrt{21}}{21}$

14. Consider a sequence of squares defined as follows: the first square S_1 has sides of length 1. The second square S_2 is obtained by doubling the sides of S_1, extending these sides outward in the cyclic pattern shown, and then joining the endpoints of these extended sides. The third square S_3 is obtained from S_2 in the same manner. If this pattern continues, find the side length of S_{100}.

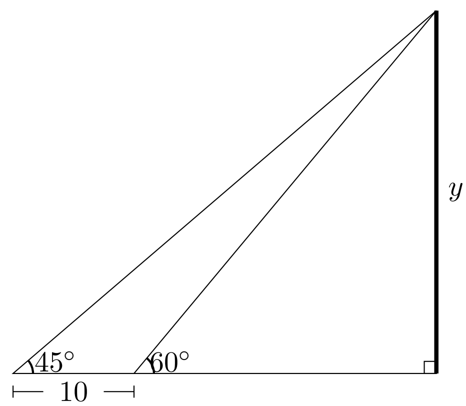


A. 2^{50} B. 2^{99} C. $2^{49}\sqrt{2}$ D. $3^{49}\sqrt{3}$ E. $5^{49}\sqrt{5}$

15. An honest coin is tossed ten times. The probability of tossing two heads and eight tails is what?

A. $\frac{45}{1024}$ B. $\frac{1}{512}$ C. $\frac{1}{5}$ D. $\frac{5}{512}$ E. None of these

16. ($\star$) Find the vertical distance y, correct to the nearest whole number.



A. $\dfrac{\sqrt{3}-1}{10\sqrt{3}}$ B. 5 C. $\dfrac{\sqrt{2}-\sqrt{6}}{20\sqrt{2}}$ D. $\dfrac{10\sqrt{3}}{\sqrt{3}-1}$ E. $\dfrac{20\sqrt{2}}{\sqrt{2}-\sqrt{6}}$

17. Find $\sin x$ if $\tan x = \dfrac{2ab}{a^2-b^2}$, where $a > b > 0$ and $0^\circ < x < 90^\circ$.

A. $\dfrac{a}{b}$ B. $\dfrac{b}{a}$ C. $\dfrac{\sqrt{a^2-b^2}}{2a}$ D. $\dfrac{\sqrt{a^2-b^2}}{2ab}$ E. $\dfrac{2ab}{a^2+b^2}$

18. ($\star$) Simplify $(1-\tan^4(\theta))\cos^2(\theta)+\tan^2(\theta)$.

A. -1 B. 0 C. 1 D. 2 E. None of these

19. George R. R. Martin is writing the sixth book in the Game of Thrones series. He numbers the book beginning with page 1. The Onion reports that to write down all of the page numbers would require 2541 digits. For example, a book with 15 pages will have page numbers $1, 2, 3, \ldots, 13, 14, 15$ for a total of 21 digits. How many pages does the book contain?

A. 880 B. 881 C. 882 D. 883 E. 884

20. What is the sum of the proper factors of 2015?

A. 36 B. 47 C. 49 D. 50 E. 2065

21. Let $x = \dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$ and $y = \dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$. Find x^2+y^2.

A. 98 B. 38 C. 0 D. -38 E. -98

22. Patrick travels from point A to point B in 8 minutes. SpongeBob travels from B to A along the same route. They start at the same time and each travels at a constant rate. If SpongeBob reaches point A 18 minutes after they meet, how many minutes did the entire trip take SpongeBob?

    A. 20 B. 22.5 C. 24 D. 25.25 E. 26

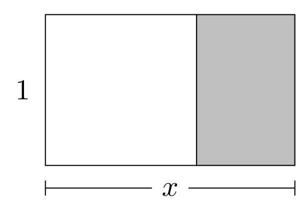
23. How many integers between 1 and 2015 are divisible by both 11 and 13 but not by 7?

    A. 12 B. 14 C. 47 D. 49 E. 51

24. If the sum of two numbers is 1 and their product is 1, then what is the sum of their cubes?

    A. 2 B. $-2-\dfrac{3\sqrt{3}i}{4}$ C. 0 D. $-\dfrac{3\sqrt{3}i}{4}$ E. -2

25. In the figure below, a rectangle of dimensions $1\times x$ has been partitioned into a square (on the left) and a second rectangle (shaded, on the right). Suppose the second rectangle is similar to the original rectangle. What is the length x?



    A. $\dfrac{1+\sqrt{5}}{2}$ B. $\dfrac{1}{2}$ C. 1 D. 2 E. $\dfrac{1+\sqrt{3}}{2}$

26. Suppose $\sin(\theta)+\cos(\theta)=\dfrac{1}{3}$. Find $\sin(\theta)\cos(\theta)$.

    A. $-\dfrac{1}{9}$ B. $-\dfrac{2}{9}$ C. $-\dfrac{4}{9}$ D. $\dfrac{1}{9}$ E. $\frac{2}{9}$

27. Find the smallest positive integer that is divisible by every integer from 1 up to 12.

    A. 4,620 B. 5,544 C. 9,240 D. 13,860 E. 27,720

28. A lot is in the shape of a right triangle. The shorter leg measures 120 meters. The hypotenuse is 40 meters longer that the length of the longer leg. How long (in meters) is the longer leg?

    A. 160 B. 240 C. 120 D. 200 E. 180

29. Consider a sequence of integers $a_1, a_2, a_3, \ldots$ where each term in the sequence is determined from the previous term according to the formula

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even} \\ 3a_n + 1 & \text{if } a_n \text{ is odd} \end{cases}$$

For example, if the first term were $a_1 = 7$, then the next five terms would be

$$a_2 = 22, \quad a_3 = 11, \quad a_4 = 34, \quad a_5 = 17, \quad a_6 = 52.$$

Supposing the first term is $a_1 = 11$, find a_{2015}.

A. 1 B. 2 C. 4 D. 8 E. Unable to determine

30. A box contains chips, each of which is red, white, or blue. The number of blue chips is at least half the number of white chips, and at most one third the number of red chips. At least 55 chips are white or blue. Find the minimum number of red chips.

A. 24 B. 57 C. 33 D. 54 E. 45

31. Which of the following is a factor of $p(x) = 4x^5 + 4x^4 - 4x^3 - 4x^2 + x + 1$?

A. $x - \frac{1}{2}$ B. $x - 1$ C. $x - \sqrt{2}$ D. $x - \frac{\sqrt{2}}{2}$ E. $x - 2\sqrt{2}$

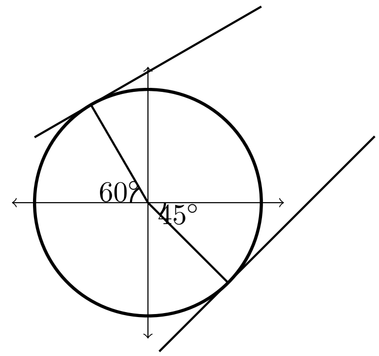
32. Compute the probability that a randomly chosen basic trigonometric function, when divided by a different randomly chosen basic trigonometric function, will result in a quotient that is itself one of the six basic trigonometric functions.

A. $\frac{1}{3}$ B. $\frac{2}{5}$ C. $\frac{7}{15}$ D. $\frac{1}{6}$ E. None of these

33. Three men, Andrew, Brian, and Connor, working together, do a job in 6 hours less time than Andrew alone, in one hour less time than Brian alone, and in one-half the time needed by Connor when working alone. Find the number of hours needed by Andrew and Brian, working together, to do the job.

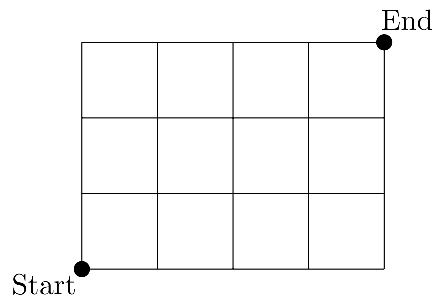
A. $\frac{5}{2}$ B. $\frac{3}{2}$ C. $\frac{4}{3}$ D. $\frac{5}{4}$ E. $\frac{3}{4}$

34. ($\star$) Pictured to the right is the unit circle of radius 1 centered at the origin. The tangent lines to two points on the circle have been drawn. These lines are not parallel, so if extended into the first quadrant they will eventually meet. Find the x-coordinate of the point where these two tangent lines meet.



A. $\dfrac{2\sqrt{3}-3\sqrt{2}}{3-\sqrt{3}}$ B. $\dfrac{2\sqrt{3}+3\sqrt{2}}{3-\sqrt{3}}$ C. $\dfrac{-2\sqrt{3}+3\sqrt{2}}{3-\sqrt{3}}$ D. $\dfrac{2\sqrt{3}+3\sqrt{2}}{3+\sqrt{3}}$ E. $\dfrac{2\sqrt{3}+3\sqrt{2}}{-3-\sqrt{3}}$

35. The figure to the right represents a 4×3 block of city streets. Assuming you can only move north and east, how many different paths are there from the starting location to the ending location?



A. 30 B. 34 C. 35 D. 36 E. 40