2022 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 6, 10, 17, 26, 32

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Suppose a, b, and c are positive real numbers. If $a+\frac{1}{b}=7c$ and $b+\frac{1}{a}=5c$, compute $\frac{a}{b}+\frac{b}{a}$.

   A. $\frac{12}{35}$ B. $\frac{74}{35}$ C. $\frac{35}{17}$ D. $\frac{17}{12}$ E. $\frac{21}{5}$

Solution: B. $\frac{74}{35}$

If $a+\frac{1}{b}=7c$, then clear denominators to obtain $ab+1=7bc$. Likewise, since $b+\frac{1}{a}=5c$, we find $ab+1=5ac$. Then $5ac=7bc$, and c cancels, leaving $b=\frac{5}{7}a$. Then

$$\frac{a}{b}+\frac{b}{a}=\frac{a}{\frac{5}{7}a}+\frac{\frac{5}{7}a}{a}=\frac{7}{5}+\frac{5}{7}=\frac{74}{35}.$$

2. An integer is chosen at random from the set $\{100,101,102,\ldots,150\}$. What is the probability that this number is prime?

   A. 0 B. $\frac{9}{50}$ C. $\frac{11}{50}$ D. $\frac{11}{51}$ E. $\frac{10}{51}$

Solution: E. $\frac{10}{51}$

Any composite number $n\leq 150$ must have a prime factor less than $\sqrt{150}\approx 12$. The only primes less than 12 are 2, 3, 5, 7, and 11. So begin with the numbers $\{100,101,\ldots,150\}$, then eliminate all multiples of 2, then all multiples of 3, then all multiples of 5, then 7, and finally 11. Any number left standing is thus prime. We find exactly ten primes between 100 and 150:

$$101,103,107,109,113,127,131,137,139,149$$

Since the set $\{100,101,102,\ldots 150\}$ has 51 elements, the probability of selecting a prime from this set is $\frac{10}{51}$.

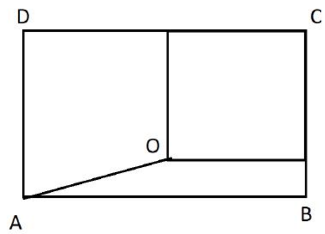
3. Suppose a and b are integers with $a>1$. Suppose the numbers $7b+6$ and $35b+11$ are both divisible by a. Compute a.

   A. 19 B. 20 C. 21 D. 22 E. 23

Solution: A. 19

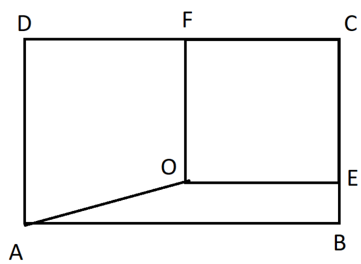
If $7b+6$ is evenly divisible by a, then $7b+6=na$ for some integer n. Likewise, $35b+11=ma$ for some integer m. Multiplying the first equation by 5 and then subtracting the second equation from it yields $19=(5n-m)a$. This means 19 is evenly divisible by the integers $5n+m$ and a. But 19 is a prime number, so it is only divisible by 1 and itself. And since $a>1$, we must have $a=19$.

4. Point O is inside a rectangle ABCD. The distance from O to A is 2, same as the distance from O to BC and from O to CD. What is the largest possible area of ABCD?



A. $3+2\sqrt{2}$ B. $12-2\sqrt{2}$ C. 10 D. $6+4\sqrt{2}$ E. $4+3\sqrt{2}$

Solution: D. $6+4\sqrt{2}$



OE is parallel to AB and OF is parallel to AD. Let $EB = x$ and $FD = y$, we have

$$x^2 + y^2 = OA^2 = 4.$$

Since $(x+y)^2 \leq 2(x^2+y^2) = 8$, we know $x + y \leq \sqrt{8}$. In addition, $xy \leq \dfrac{x^2+y^2}{2} = 2$.

Therefore,

$$Area(ABCD) = (2+x)(2+y) = 4 + 2(x+y) + xy \leq 6 + 4\sqrt{2}.$$

5. A careless mail carrier has four letters addressed to four different houses. They randomly place one letter into each house's mailbox. What is the probability that every letter is placed in the correct mailbox?

A. $\frac{1}{4}$ B. $\frac{1}{6}$ C. $\frac{1}{10}$ D. $\frac{1}{24}$ E. $\frac{1}{100}$

Solution: D. $\dfrac{1}{24}$

There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to place the letters into the mailboxes (4 choices for which letter goes into the first mailbox, then 3 choices for the second, etc.). Only one of those ways leads to every letter in its correct mailbox, so the probability of this outcome is $\dfrac{1}{24}$.

6. ($\star$) How many distinct prime factors does the number 2022^{2022} have?

A. 0 B. 3 C. 18 D. 175 E. 2022

Solution: B. 3

Start with $2022 = 2 \cdot 1011$. Then note 1011 is divisible by 3 (because its sum of digits is divisible by 3). We find that $1011 = 3 \cdot 337$. Now 337 is prime: since $\sqrt{337} < \sqrt{400} = 20$, we can confirm 337 is prime by attempting to divide it by all prime numbers less than 20 and observing that each leaves a nonzero remainder. So 2022 has exactly three prime factors, and the prime factorization of 2022 is

$$2022 = 2 \cdot 3 \cdot 337.$$

But then 2022^{2022} will have the same three prime factors as well! The prime factorization of 2022^{2022} is

$$2022^{2022} = 2^{2022}3^{2022}337^{2022}.$$

7. Suppose x is a real number. If $x^2 + 7x + 11 = 12$ and $x < 0$, compute x^3.

A. $-130 + 18\sqrt{53}$ B. $-37(53)^{3/2}$ C. $-100 + \sqrt{53}$ D. $-227 - 14\sqrt{53}$ E. $-182 - 25\sqrt{53}$

Solution: E. $-182 - 23\sqrt{53}$

If $x^2 + 7x + 11 = 12$, then $x^2 + 7x - 1 = 0$. The quadratic formula gives us

$$x = \frac{-7 \pm \sqrt{53}}{2}.$$

Since we need $x < 0$, we must choose $x = \dfrac{-7 - \sqrt{53}}{2}$, and so

$$\begin{aligned} x^3 = \frac{(-7-\sqrt{53})^3}{2^3} &= \frac{1}{8}\left[(-7)^3 + 3(-7)^2(-\sqrt{53}) + 3(-7)(-\sqrt{53})^2 + (-\sqrt{53})^3\right] \\ &= \frac{1}{8}\left[-343 - 147\sqrt{53} - 21(53) - 53\sqrt{53}\right] \\ &= \frac{1}{8}\left[-1456 - 200\sqrt{53}\right] \\ &= -182 - 25\sqrt{53}. \end{aligned}$$

8. Let x, y, z be real numbers satisfying the following:

$$\begin{aligned} \frac{1}{x} + \frac{1}{y+z} &= \frac{1}{2021} \\ \frac{1}{y} + \frac{1}{x+z} &= \frac{1}{2022} \\ \frac{1}{z} + \frac{1}{x+y} &= \frac{1}{2023} \end{aligned}$$

What is the value of $\dfrac{xy}{x+y+z}$?

A. 674 B. 1010 C. 1011 D. 2022 E. 2023

Solution: B. 1010

From the assumptions, we find

$$\frac{xy+xz}{x+y+z} = 2021 \tag{1}$$
$$\frac{xy+yz}{x+y+z} = 2022 \tag{2}$$
$$\frac{xz+yz}{x+y+z} = 2023 \tag{3}$$

Adding all three together, we will get

$$\frac{xy+yz+xz}{x+y+z} = 3033.$$

Now subtract equation (3) from the above result,

$$\frac{xy}{x+y+z} = 3033 - 2023 = 1010.$$

9. Compute $\log_2 2022$ to the nearest whole number.

A. 8 B. 9 C. 10 D. 11 E. 12

Solution: D. 11

Note 2022 is between the powers $2^{10} = 1024$ and $2^{11} = 2048$. Since 2022 is much closer in value to 2048 than it is to 1024, we use the approximation $2022 \approx 2048$. Thus $\log_2 2022 \approx \log_2(2^{11}) = 11\log_2 2$. Since $\log_2 2 = 1$, we get

$$\log_2 2022 \approx 11.$$

10. ($\star$) A boy saves his money in a very large piggy bank.

He places:
1 penny into the bank on day #1,
3 pennies into the bank on day #2 (i.e. $2+1$)
6 pennies on day #3 (i.e. $3+2+1$)
10 pennies on day #4 (i.e. $4+3+2+1$)
... and so on.

Thus, after the 1st four days the bank contains 20 pennies.

How many pennies are in the bank after the first sixty days?

A. 33,640 B. 36,900 C. 37,820 D. 39,960 E. 42,180

Solution: C. 37,820

The number of pennies in the bank after n days, T_n, is given by $T_n = \sum_{i=1}^{n}\sum_{j=1}^{i} j$.

Now $\sum_{j=1}^{i} j = \frac{i(i+1)}{2}$.

So $T_n = \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\sum_{i=1}^{n}(i^2+i) = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i = \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\cdot\frac{n(n+1)}{2}$.

Setting $n = 60$, we get $T_{60} = \frac{1}{2}\cdot\frac{60(61)(121)}{6} + \frac{1}{2}\cdot\frac{60(61)}{2} = \frac{1}{2}\cdot 73,810 + \frac{1}{2}\cdot 1,830 = 37,820$.

11. When 2022^{2021} is divided by the sum of 21 and 22, what is the remainder?

A. 1 B. 3 C. 20 D. 21 E. 22

Solution: A. 1

Write 2022 as $2021 + 1$. We are trying to evaluate $(2021+1)^{2021} \mod 43$. Using the binomial theorem, we see that all terms in this expansion are divisible by 2021 except for the very last term. In addition, 2021 is divisible by 43. Therefore $(2021+1)^{2021} \mod 43 = 1^{2021} = 1$.

12. For how many odd numbers n is $n^2 - 8n + 17$ a prime number?

A. 0 B. 1 C. 2 D. 3 E. Infinitely many

Solution: C. 2

Since n is odd, n^2 is also an odd number. $n^2 - 8n + 17$ is therefore an even number. The only even prime number is 2. This implies

$$n^2 - 8n + 17 = 2.$$

13. The cost of making a pair of shoes is m dollars and the original sales price is $a\%$ higher. If the manager decided to raise the price by $b\%$, what is the new price?

A. $m(a\%)(1+b\%)$ B. $m(1+a\%)(1+b\%)$ C. $m(1+a\%)(b\%)$ D. $m(a\%)(b\%)$ E. $m(1+a\%)(1-b\%)$

Solution: B. $m(1+a\%)(1+b\%)$

We know the original sales price is $m(1+a\%)$.

Now it is raised by $b\%$, the new price is $m(1+a\%)(1+b\%)$.

14. A furniture company employs three shifts of workers to manufacture their tables.

- Shift #1 produces 50% of all tables
- Shift #2 produces 30% of all tables
- Shift #3 produces 20% of all tables

Historical data shows:

- 20% of the tables produced by Shift #1 have at least one defect
- 10% of the tables produced by Shift #2 have at least one defect

- 30% of the tables produced by Shift #3 have at least one defect

A randomly selected table has no defects.

The probability that this table was produced by Shift #1 is closest to?

A. 35% B. 40% C. 45% D. 50% E. 55%

Solution: D. 50%

We are looking for the probability that the table was produced during shift 1 given that the table is not defective. The numerator of the probability we want is the probability that table is both produced during shift 1 and is not defective, which is $(0.5)(0.8)$. The denominator is the probability that the table is not defective, and taking all shifts into consideration, this is $(0.5)(0.8)+(0.3)(0.9)+(0.2)(0.7)$.

Thus, we find

$P(\text{Produced by shift \#1}) = \dfrac{(0.5)(0.8)}{(0.5)(0.8)+(0.3)(0.9)+(0.2)(0.7)} = \dfrac{40}{81}$, which is about 49%.

15. A box contains 5 marbles – one marble each of the following five colors: red, white, blue, green, orange. Four marbles will be selected, with replacement, and order does not matter.

    One possible outcome is: The blue marble is selected twice, the red once and the orange once.

    Another possible outcome is: The blue marble is selected all four times.

    How many distinct possible outcomes are there?

    A. 15 B. 30 C. 70 D. 625 E. None of these

Solution: C. 70

When selecting r times from a set of n distinct objects, with replacement, unordered, the counting formula is:

$$n(S) = \binom{n+r-1}{r}, \text{ which becomes } n(S) = \binom{5+4-1}{4} = \binom{8}{4} = 70.$$

16. In the game of Yahtzee, players take turns rolling five fair dice.

    "Two Pair" is rolled when one number appears exactly twice, another number appears exactly twice, and a third number appears exactly once. For example: $(2,2,4,5,5)$.

    Given that a roll does not result in the number 6 being rolled on any die, what is the probability that the roll results in Two Pair?

    A. $\dfrac{25}{108}$ B. $\dfrac{36}{125}$ C. $\dfrac{39}{108}$ D. $\dfrac{25}{54}$ E. $\dfrac{72}{125}$

Solution: B. $\dfrac{36}{125}$

Think ordered $5-$tuples. Then the sample space consists of 6^5 equally likely outcomes.

Define the following two events: A: No 6; and B: Two Pair. We want $P(A|B) = \dfrac{P(A \cap B)}{P(A)}$.

The number of possible outcomes resulting in event A is 5^5.

The number of outcomes is $A \cap B$ has the following components:

First, choose 3 distinct numbers without 6, There are $\binom{5}{3} = 10$ of these.

Next, choose 2 of these to be pairs. There are $\binom{3}{2} = 3$ pairs.

Since order matters, there are $\dfrac{5!}{2!2!1!} = 30$ permutations.

Thus, there are $10 \cdot 3 \cdot 30 = 900$ possible outcomes in $A \cap B$.

Finally, we have $P(A|B) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{900/6^5}{5^5/6^5} = \dfrac{900}{5^5} = \dfrac{36}{125}$.

17. ($\star$) An infinite geometric series has first term equal to 4 and common ratio equal to $\dfrac{2}{3}$. What is the sum of the first 50 even numbered terms?

Note: The first even numbered term is $4 \cdot \dfrac{2}{3}$.

A. $\dfrac{24}{5}\left(1 - \left(\dfrac{4}{9}\right)^{50}\right)$ B. $\dfrac{24}{5}\left(1 - \left(\dfrac{4}{9}\right)^{51}\right)$ C. $\dfrac{54}{5}\left(\dfrac{4}{9} - \left(\dfrac{4}{9}\right)^{50}\right)$ D. $\dfrac{54}{5}\left(1 - \left(\dfrac{2}{3}\right)^{100}\right)$

E. None of these

Solution: A. $\dfrac{24}{5}\left(1 - \left(\dfrac{4}{9}\right)^{50}\right)$

The terms of the series are $4\left(\dfrac{2}{3}\right)^{k-1}$, for $k = 1, 2, \ldots$

The sum of the first 50 even numbered terms is $S = \displaystyle\sum_{i=1}^{50} 4\left(\frac{2}{3}\right)^{2i-1}$.

Factoring out the constants, we get $S = 4\left(\dfrac{2}{3}\right)^{-1} \displaystyle\sum_{i=1}^{50} \left(\frac{4}{9}\right)^{i}$.

Thus, $S = 6 \cdot \dfrac{\left(\frac{4}{9} - \left(\frac{4}{9}\right)^{51}\right)}{1 - \frac{4}{9}} = 6\left(\dfrac{4}{5}\right)\left(1 - \left(\dfrac{4}{9}\right)^{50}\right) = \left(\dfrac{24}{5}\right)\left(1 - \left(\dfrac{4}{9}\right)^{50}\right)$.

18. A fair six-sided die is tossed five consecutive times. The probability that exactly three of the five tosses result in the same number is closest to which percent?

A. 5% B. 10% C. 15% D. 20% E. 25%

Solution: D. 20%

There are 6^5 possible outcomes to rolling a six-sided die five times.

To get exactly three of the same number there are two cases.

Case 1: The two other rolls are distinct numbers.

Step 1: Choose the three numbers: $\binom{6}{3} = 20$ ways.

Step 2: Decide which of these three is the number repeated 3 times: $\binom{3}{1} = 3$ ways

Step 3: Run the possible permutations of the 5 rolls: $\frac{5!}{3!} = 20$ ways.

So there are $20 \times 3 \times 20 = 1200$ possibilities.

Case 2: The two other rolls are the same number.

Step 1: Choose the two numbers: $\binom{6}{2} = 15$ ways.

Step 2: Decide which of these two is the number repeated 3 times: $\binom{2}{1} = 2$ ways

Step 3: Run the possible permutations of the 5 rolls: $\frac{5!}{3!2!} = 10$ ways.

So there are $15 \times 2 \times 10 = 300$ possibilities.

Thus, to get 3 of a kind, we have 1500 possibilities. So our probability is $\frac{1500}{6^5} \approx 19.3\%$, which is closest to 20%.

19. Suppose 14 people order dessert. Everyone orders at least one kind of dessert, no one orders more than one of the same kind of dessert, and no one orders exactly two different kinds of dessert. Five order brownies, six order cookies, and seven order milkshakes. How many people order all three kinds of dessert?

A. 5 B. 4 C. 3 D. 2 E. 1

Solution: D. 2

The number of desserts ordered is $5 + 6 + 7 = 18$. Since all 14 people ordered at least one dessert and no one ordered exactly two, the number of desserts each person got is either 1 or at least 3. Since there are only three kinds of desserts and no one ordered more than one of any kind, no one got more than 3 desserts. Thus each of the 14 people got 1 or 3 desserts for a total of 18 desserts. this means that two people got 3 desserts and 12 people got 1 dessert.

20. Compute $\dfrac{1}{\sqrt{170} - \sqrt{169}}$ to the nearest integer.

A. 25 B. 26 C. 27 D. 28 E. None of these

Solution: B. 26

Multiplying $\dfrac{1}{\sqrt{170}-\sqrt{169}}$ by the algebraic conjugate in the numerator and denominator gives $\dfrac{1}{\sqrt{170}-\sqrt{169}} \cdot \dfrac{\sqrt{170}+\sqrt{169}}{\sqrt{170}+\sqrt{169}} = \dfrac{\sqrt{170}+\sqrt{169}}{170-169} = \dfrac{\sqrt{170}+13}{1}$.

Now, $13^2 = 169$ and $14^2 = 196$, so $13 < \sqrt{170} < 14$ but $\sqrt{170}$ is very close to 13.

Thus, $\dfrac{1}{\sqrt{170}-\sqrt{169}} \approx 13 + 13 = 26$.

21. Three different clubs at school each took part in a math contest that was graded out of 100. The overall average score for everyone who took the exam from the three clubs was an 82.

For each club the average score was as follows:
For the 8 members of Club A, the average was 83,
for the 5 members of Club B, the average was 87,
for the x members of Club C, the average was 79. What is the value of x?

A. 8 B. 9 C. 10 D. 11 E. 12

Solution: D. 11

The sum of the scores for Club A is $8(83) = 664$.
The sum of the scores for Club B is $5(87) = 435$.
The sum of the scores for Club C is $x(79) = 79x$.

Thus the total average is $\dfrac{664 + 435 + 79x}{8 + 5 + x} = 82$.

So $1099 + 79x = 82(13) + 82x$, which simplifies to $33 = 3x$. Thus $x = 11$.

Hence, $x = \dfrac{1,099 - (820 + 246)}{3} = \dfrac{1,099 - 1,066}{3} = \dfrac{33}{3} = 11$.

22. Find all real values of x that satisfy the inequality $\dfrac{(x-2)^{1/3}(2x+3)^2}{(x+5)^3(x^2+4)} \geq 0$.

A. $(-\infty, -5) \cup [2, \infty)$ B. $(-\infty, -2) \cup \{-3/2\} \cup (2, \infty)$ C. $\{-3/2\} \cup [2, \infty)$
D. $(-\infty, -5) \cup \{-3/2\} \cup [2, \infty)$ E. $(-\infty, -5) \cup \{-3/2\} \cup [8, \infty)$

Solution: D. $(-\infty, -5) \cup \{-3/2\} \cup [2, \infty)$

Let $f(x) = \dfrac{(x-2)^{1/3}(2x+3)^2}{(x+5)^3(x^2+4)}$. First, note that $f(-5)$ is undefined and $f(x) = 0$ when $x = 2, -3/2$. Thus, it suffices to check whether f is positive or negative at some value between each of the critical points. This gives $f(x) \geq 0$ on $(-\infty, -5) \cup \{-3/2\} \cup [2, \infty)$.

23. Let $x_1, x_2, \dots x_{2022}$ be integers such that:

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1 \\ x_2 & + & x_3 & = & 2 \\ & & & \vdots & \\ x_{2021} & + & x_{2022} & = & 2021 \end{array}$$

If $x_1 + x_{499} + x_{999} + x_{1501} = 222$, what is x_{2022}?

A. 1010 B. 1329 C. 1330 D. 1331 E. None of these

Solution: C. 1330

We have $x_{i+1} + x_{i+2} = i + 1$ and $x_i + x_{i+1} = i$. Subtracting we see that $x_{i+2} - x_i = 1$. So, $x_1, x_3, \ldots, x_{2021}$ form an arithemetic sequence with common difference of 1.

Now $\dfrac{x_1 + x_{1501}}{2} = x_{751}$ and $\dfrac{x_{499} + x_{999}}{2} = x_{749}$. So, $x_1 + x_{1501} = 2x_{751}$ and $x_{499} + x_{999} = 2x_{749}$. Then

$$x_1 + x_{499} + x_{999} + x_{1501} = 2x_{751} + 2x_{749} = 222.$$

Therefore $x_{751} + x_{749} = 111$.

Then, since $x_{751} - x_{749} = 1$, add this to the previous equation to get $2x_{751} = 112$. So $x_{751} = 56$.

Now $x_{2021} = x_{751} + \dfrac{2021 - 751}{2} = 56 + 635 = 691$. Since $x_{2021} + x_{2022} = 2021, x_{2022} = 1330$.

24. A Kangaroo is showing off her jumping skills. Her first jump she jumps 1 meter. On each successive jump, she hops twice as far as her previous jump. So she jumps 2 meters on her second jump and 4 meters on her third. How far does she jump on her 9^{th} jump?

A. 14 B. 64 C. 128 D. 256 E. 512

Solution: D. 256

So she jumps: $2^0, 2^1, 2^2, 2^3 \ldots$. On the 9^{th} jump she jumps $2^8 = 256$

25. For how many positive integers x less than 2022 is $x^3 - x^2 + x - 1$ prime?

A. 0 B. 1 C. 2 D. 3 E. 4

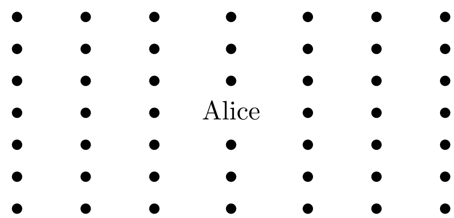
Solution: B. 1

We have $x^3 - x^2 + x - 1 = (x^2 + 1)(x - 1)$. Now x is a postive integer. If $x > 2$, then $x - 1 > 1$ and $x^2 + 1 > 1$, so it cannot be prime. This leaves us to check when $x = 1$ and $x = 2$.

When $x = 1$, $x^3 - x^2 + x - 1 = 1 - 1 + 1 - 1 = 0$ is not prime.

When $x = 2$, $x^3 - x^2 + x - 1 = 8 - 4 + 2 - 1 = 5$ which is prime. So there is only one.

26. ($\star$) Alice is at the middle of the 7×7 lattice below. At every second, with a probability of $\dfrac{1}{4}$, she moves left, right, up, or down. Let $\dfrac{a}{b}$ be the probability she ends up back in the middle after 4 seconds (Note: let a and b be postive and in lowest terms). What is $b - a$?



A. 55 B. 57 C. 59 D. 63 E. None of these

Solution: A. 55

The probability of any given path Alice can move is $\left(\frac{1}{4}\right)^4 = \frac{1}{256}$.

To end up back at the middle there are three options.

- She moves only right or left. In this case she must move left exactly twice and right exactly twice. So in four moves she must move left twice, which she can do in $\binom{4}{2} = 6$ ways.
- She moves only up or down. In this case she must move up exactly twice and down exactly twice. So in four moves she must move up twice, which she can do in $\binom{4}{2} = 6$ ways.
- She moves **both** up/down and left/right. To end up back in the middle she must move in each direction exactly once. So there are $4! = 24$ ways to do this.

So in total Alice can end up back in the middle in $6 + 6 + 24 = 36$ ways. So the probability she is back in the middle is $\frac{36}{256} = \frac{9}{64}$. And $b - a = 64 - 9 = 55$

27. There are six men that need to be lined up in a single line such that no man is standing between two other men taller than himself. How many ways can they line up?

A. 30 B. 32 C. 48 D. 64 E. 120

Solution: B. 32

Start with the tallest man, m_1. Take the second tallest man m_2 and have him stand either to the left or right of m_1, so two options. Now take the third tallest man, m_3 and have him stand either to the left or right of the pair of m_1 and m_2, so two options. Continue this until we get the six tallest (overall shortest man) who can either be on the left end or the right end, so two options. In total there are $2^5 = 32$ possible lines.

28. A basket contains 3 red marbles and 4 blue marbles. Suppose your friend randomly picks two marbles from the basket simultaneously (without replacement) and without looking. Find the probability that the selected marbles are different colors. Assume that each marble has an equal chance of being chosen and there is no way to distinguish between marbles without looking at them.

A. $\frac{4}{7}$ B. $\frac{5}{9}$ C. $\frac{8}{13}$ D. $\frac{2}{5}$ E. $\frac{6}{11}$

Solution: A. $\frac{4}{7}$

There are $\binom{7}{2} = \frac{7!}{2!5!} = \frac{7 \cdot 6}{2} = 21$ ways to pick 2 marbles from the basket.

There are $\binom{3}{1} = 3$ ways to pick one red marble and $\binom{4}{1} = 4$ ways to pick a blue marble.

Thus, the desired probability is $\frac{12}{21} = \frac{4}{7}$.

29. Find the vertex of the parabola whose equation is $y = -3x^2 - 2x + 1$.

A. $\left(-\frac{1}{3}, -\frac{4}{3}\right)$ B. $\left(\frac{1}{3}, \frac{4}{3}\right)$ C. $\left(-\frac{1}{3}, \frac{4}{3}\right)$ D. $\left(-\frac{1}{3}, 1\right)$ E. None of these

Solution: C. $\left(-\frac{1}{3}, \frac{4}{3}\right)$

Note that $y = -3x^2 - 2x + 1 = -3\left(x^2 + \frac{2x}{3}\right) + 1 = -3\left(x^2 + \frac{2x}{3} + \frac{1}{9} - \frac{1}{9}\right) + 1$

$= -3\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right] + 1 = -3\left(x + \frac{1}{3}\right)^2 + \frac{1}{3} + 1 = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}$.

The vertex is $x = -\frac{1}{3}$, $y = \frac{4}{3}$.

30. Suppose $\sec\theta = \frac{x}{y}$. Find $\cos(2\theta)$.

A. $\frac{2x^2 - y^2}{x^2}$ B. $\frac{2x^2 - y^2}{y^2}$ C. $\frac{2y^2 - x^2}{x^2}$ D. $\frac{2y^2 - x^2}{y^2}$ E. None of these

Solution: C. $\frac{2y^2 - x^2}{x^2}$

Since $\sec\theta = \frac{x}{y}$, $\cos\theta = \frac{y}{x}$. So $\cos^2(\theta) = \frac{y^2}{x^2}$

Using our double angle identity, we have $\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$. So

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 2\frac{y^2}{x^2} - 1 = \frac{2y^2 - x^2}{x^2}$$

31. Consider the functions $f(x) = x^2 - 1, g(x) = \cos(x)$, and $h(x) = 2x + 3$. Compute $h \circ (g \circ f)(x)$.

A. $2\cos(x^2 - 1) + 3$ B. $(2x + 3)\cos(x)(x^2 - 1)$ C. $\cos^2(2x + 3) - 1$ D. $2x + 3\cos(x^2 - 1)$
E. $\sin(2x - 1) + 3$

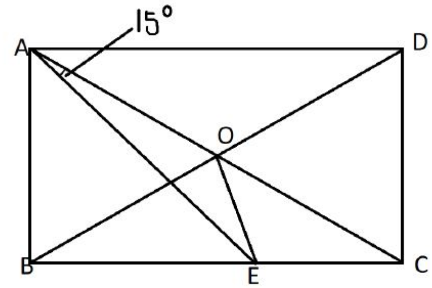
Solution: A. $2\cos(x^2 - 1) + 3$

We have

$$h \circ (g \circ f)(x) = h(g(f(x)))$$

$$= h(g(x^2-1)) = h(\cos(x^2-1)) = 2\cos(x^2-1)+3$$

32. ($\star$) ABCD is a rectangle. Point O is the intersection of two diagonals BD and AC. E is a point on BC so that AE is the bisector of $\angle BAD$. Knowing $\angle EAO$ is 15^o, what is the angle measure of BOE?



A. 60^o B. 65^o C. 70^o D. 75^o E. 80^o

Solution: D. 75^o

$\angle BAE$ is 45^o. Therefore ABE is an isosceles right triangle and $AB = BE$.

Next, we know $\angle CAD = 30^o$ since $\angle EAO = 15^o$. This implies $\angle ACD = \angle ABO = 60^o$.

As $\angle BAO$ is also 60^o, we find that ABO is an equilateral triangle. Therefore, BO=AB. This means BOE is an isosceles triangle too.

We know $\angle OBE = \angle CAD = 30^o$. Therefore $\angle BOE = (180-30)/2 = 75^o$.

33. Given that $x = 8756(a-b)$, $y = 8756(b-c)$, and $z = 8756(c-a)$, compute $\dfrac{x^2+y^2+z^2}{xy+yz+xz}$ if $xy+yz+xz \neq 0$.

A. -2 B. -1 C. 0 D. 1 E. 2

Solution: A. -2
Adding the three equations gives $x + y + z = 8756a – 8756b + 8756b – 8756c + 8756c – 8756a = 0$. Squaring both sides of $x+y+z=0$ gives $x^2+y^2+z^2+2(xy+xz+yz)=0$. Thus, $x^2+y^2+z^2 = -2(xy+xz+yz)$. Since $xy+yz+xz \neq 0$, we can divide both sides by this quantity, yielding the desired $\dfrac{x^2+y^2+z^2}{xy+yz+xz} = -2$.

34. Let α and β be the roots of the equation,

$$x^2 + px + 1 = 0,$$

and let γ and δ be the roots of the equation,

$$x^2 + qx + 1 = 0.$$

What is the value of the expression $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$?

A. $p+q$ B. q^2-p^2 C. p^2q^2 D. $(p+q)^2$ E. pq^2+qp^2

Solution:
B. $q^2 - p^2$

Using the relations between the roots and the coefficients of a quadratic equation, we know $\alpha + \beta = -p$, $\alpha\beta = 1$, $\gamma + \delta = -q$, and $\gamma\delta = 1$. From these we thus obtain,

$$\begin{aligned}(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta) &= [(\alpha-\gamma)(\beta+\delta)][(\beta-\gamma)(\alpha+\delta)] \\ &= (\alpha\beta+\alpha\delta-\beta\gamma-\gamma\delta)(\alpha\beta+\beta\delta-\alpha\gamma-\gamma\delta) \\ &= (\alpha\delta-\beta\gamma)(\beta\delta-\alpha\gamma) \\ &= \alpha\beta\delta^2-\alpha^2\gamma\delta-\beta^2\gamma\delta+\alpha\beta\gamma^2 \\ &= \delta^2-\alpha^2-\beta^2+\gamma^2 \\ &= [(\delta-\gamma)^2+2\delta\gamma]-[(\alpha+\beta)^2-2\alpha\beta] \\ &= (q^2-2)-(p^2-2) \\ &= q^2-p^2\end{aligned}$$

35. If $\log_2 \pi = a$ and $\log_5 \pi = b$, then what is $\log_{10} \pi$?

A. ab B. a/b C. $ab/(a+b)$ D. $1/a + 1/b$ E. $a+b$

Solution: C. $ab/(a+b)$

By the rules of logarithms, we have $2^a = \pi$ and $5^b = \pi$, from which we find $\pi^{1/a} = 2$ and $\pi^{1/b} = 5$ respectively. Thus $\pi^{1/a+1/b} = 10$ which yields $10^{ab/(a+b)} = \pi$.