2022 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 6, 10, 17, 26, 32

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Suppose a, b, and c are positive real numbers. If $a + \dfrac{1}{b} = 7c$ and $b + \dfrac{1}{a} = 5c$, compute $\dfrac{a}{b} + \dfrac{b}{a}$.

   A. $\dfrac{12}{35}$ B. $\dfrac{74}{35}$ C. $\dfrac{35}{17}$ D. $\dfrac{17}{12}$ E. $\dfrac{21}{5}$

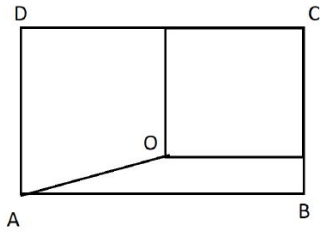
2. An integer is chosen at random from the set $\{100, 101, 102, \ldots, 150\}$. What is the probability that this number is prime?

   A. 0 B. $\dfrac{9}{50}$ C. $\dfrac{11}{50}$ D. $\dfrac{11}{51}$ E. $\dfrac{10}{51}$

3. Suppose a and b are integers with $a > 1$. Suppose the numbers $7b + 6$ and $35b + 11$ are both divisible by a. Compute a.

   A. 19 B. 20 C. 21 D. 22 E. 23

4. Point O is inside a rectangle ABCD. The distance from O to A is 2, same as the distance from O to BC and from O to CD. What is the largest possible area of ABCD?



   A. $3 + 2\sqrt{2}$ B. $12 - 2\sqrt{2}$ C. 10 D. $6 + 4\sqrt{2}$ E. $4 + 3\sqrt{2}$

5. A careless mail carrier has four letters addressed to four different houses. They randomly place one letter into each house's mailbox. What is the probability that every letter is placed in the correct mailbox?

   A. $\dfrac{1}{4}$ B. $\dfrac{1}{6}$ C. $\dfrac{1}{10}$ D. $\dfrac{1}{24}$ E. $\dfrac{1}{100}$

6. ($\star$) How many distinct prime factors does the number 2022^{2022} have?

   A. 0 B. 3 C. 18 D. 175 E. 2022

7. Suppose x is a real number. If $x^2 + 7x + 11 = 12$ and $x < 0$, compute x^3.

   A. $-130 + 18\sqrt{53}$ B. $-37(53)^{3/2}$ C. $-100 + \sqrt{53}$ D. $-227 - 14\sqrt{53}$ E. $-182 - 25\sqrt{53}$

8. Let x, y, z be real numbers satisfying the following:

$$\frac{1}{x} + \frac{1}{y+z} = \frac{1}{2021}$$
$$\frac{1}{y} + \frac{1}{x+z} = \frac{1}{2022}$$
$$\frac{1}{z} + \frac{1}{x+y} = \frac{1}{2023}$$

   What is the value of $\dfrac{xy}{x+y+z}$?

   A. 674 B. 1010 C. 1011 D. 2022 E. 2023

9. Compute $\log_2 2022$ to the nearest whole number.

   A. 8 B. 9 C. 10 D. 11 E. 12

10. ($\star$) A boy saves his money in a very large piggy bank.

    He places:
    1 penny into the bank on day #1,
    3 pennies into the bank on day #2 (i.e. $2+1$)
    6 pennies on day #3 (i.e. $3+2+1$)
    10 pennies on day #4 (i.e. $4+3+2+1$)
    ... and so on.

    Thus, after the 1st four days the bank contains 20 pennies.

    How many pennies are in the bank after the first sixty days?

    A. 33,640 B. 36,900 C. 37,820 D. 39,960 E. 42,180

11. When 2022^{2021} is divided by the sum of 21 and 22, what is the remainder?

    A. 1 B. 3 C. 20 D. 21 E. 22

12. For how many odd numbers n is $n^2 - 8n + 17$ a prime number?

    A. 0 B. 1 C. 2 D. 3 E. Infinitely many

13. The cost of making a pair of shoes is m dollars and the original sales price is $a\%$ higher. If the manager decided to raise the price by $b\%$, what is the new price?

    A. $m(a\%)(1+b\%)$ B. $m(1+a\%)(1+b\%)$ C. $m(1+a\%)(b\%)$ D. $m(a\%)(b\%)$
    E. $m(1+a\%)(1-b\%)$

14. A furniture company employs three shifts of workers to manufacture their tables.

    - Shift #1 produces 50% of all tables
    - Shift #2 produces 30% of all tables
    - Shift #3 produces 20% of all tables

    Historical data shows:

    - 20% of the tables produced by Shift #1 have at least one defect
    - 10% of the tables produced by Shift #2 have at least one defect
    - 30% of the tables produced by Shift #3 have at least one defect

    A randomly selected table has no defects.

    The probability that this table was produced by Shift #1 is closest to?

    A. 35% B. 40% C. 45% D. 50% E. 55%

15. A box contains 5 marbles – one marble each of the following five colors: red, white, blue, green, orange. Four marbles will be selected, with replacement, and order does not matter.

    One possible outcome is: The blue marble is selected twice, the red once and the orange once.

    Another possible outcome is: The blue marble is selected all four times.

    How many distinct possible outcomes are there?

    A. 15 B. 30 C. 70 D. 625 E. None of these

16. In the game of Yahtzee, players take turns rolling five fair dice.

    "Two Pair" is rolled when one number appears exactly twice, another number appears exactly twice, and a third number appears exactly once. For example: $(2, 2, 4, 5, 5)$.

    Given that a roll does not result in the number 6 being rolled on any die, what is the probability that the roll results in Two Pair?

    A. $\frac{25}{108}$ B. $\frac{36}{125}$ C. $\frac{39}{108}$ D. $\frac{25}{54}$ E. $\frac{72}{125}$

17. ($\star$) An infinite geometric series has first term equal to 4 and common ratio equal to $\frac{2}{3}$. What is the sum of the first 50 even numbered terms?

    Note: The first even numbered term is $4 \cdot \frac{2}{3}$.

    A. $\frac{24}{5}\left(1 - \left(\frac{4}{9}\right)^{50}\right)$ B. $\frac{24}{5}\left(1 - \left(\frac{4}{9}\right)^{51}\right)$ C. $\frac{54}{5}\left(\frac{4}{9} - \left(\frac{4}{9}\right)^{50}\right)$ D. $\frac{54}{5}\left(1 - \left(\frac{2}{3}\right)^{100}\right)$
    E. None of these

18. A fair six-sided die is tossed five consecutive times. The probability that exactly three of the five tosses result in the same number is closest to which percent?

    A. 5% B. 10% C. 15% D. 20% E. 25%

19. Suppose 14 people order dessert. Everyone orders at least one kind of dessert, no one orders more than one of the same kind of dessert, and no one orders exactly two different kinds of dessert. Five order brownies, six order cookies, and seven order milkshakes. How many people order all three kinds of dessert?

    A. 5 B. 4 C. 3 D. 2 E. 1

20. Compute $\frac{1}{\sqrt{170} - \sqrt{169}}$ to the nearest integer.

    A. 25 B. 26 C. 27 D. 28 E. None of these

21. Three different clubs at school each took part in a math contest that was graded out of 100. The overall average score for everyone who took the exam from the three clubs was an 82.

    For each club the average score was as follows:
    For the 8 members of Club A, the average was 83,
    for the 5 members of Club B, the average was 87,
    for the x members of Club C, the average was 79. What is the value of x?

    A. 8 B. 9 C. 10 D. 11 E. 12

22. Find all real values of x that satisfy the inequality $\frac{(x-2)^{1/3}(2x+3)^2}{(x+5)^3(x^2+4)} \geq 0$.

    A. $(-\infty, -5) \cup [2, \infty)$ B. $(-\infty, -2) \cup \{-3/2\} \cup (2, \infty)$ C. $\{-3/2\} \cup [2, \infty)$
    D. $(-\infty, -5) \cup \{-3/2\} \cup [2, \infty)$ E. $(-\infty, -5) \cup \{-3/2\} \cup [8, \infty)$

23. Let $x_1, x_2, \ldots x_{2022}$ be integers such that:

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1 \\ x_2 & + & x_3 & = & 2 \\ & & & \vdots & \\ x_{2021} & + & x_{2022} & = & 2021 \end{array}$$

If $x_1 + x_{499} + x_{999} + x_{1501} = 222$, what is x_{2022}?

A. 1010 B. 1329 C. 1330 D. 1331 E. None of these

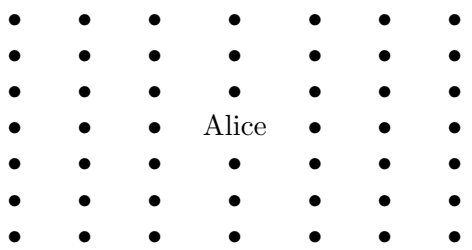
24. A Kangaroo is showing off her jumping skills. Her first jump she jumps 1 meter. On each successive jump, she hops twice as far as her previous jump. So she jumps 2 meters on her second jump and 4 meters on her third. How far does she jump on her 9^{th} jump?

A. 14 B. 64 C. 128 D. 256 E. 512

25. For how many positive integers x less than 2022 is $x^3 - x^2 + x - 1$ prime?

A. 0 B. 1 C. 2 D. 3 E. 4

26. ($\star$) Alice is at the middle of the 7×7 lattice below. At every second, with a probability of $\frac{1}{4}$, she moves left, right, up, or down. Let $\frac{a}{b}$ be the probability she ends up back in the middle after 4 seconds (Note: let a and b be postive and in lowest terms). What is $b - a$?



A. 55 B. 57 C. 59 D. 63 E. None of these

27. There are six men that need to be lined up in a single line such that no man is standing between two other men taller than himself. How many ways can they line up?

A. 30 B. 32 C. 48 D. 64 E. 120

28. A basket contains 3 red marbles and 4 blue marbles. Suppose your friend randomly picks two marbles from the basket simultaneously (without replacement) and without looking. Find the probability that the selected marbles are different colors. Assume that each marble has an equal chance of being chosen and there is no way to distinguish between marbles without looking at them.

A. $\frac{4}{7}$ B. $\frac{5}{9}$ C. $\frac{8}{13}$ D. $\frac{2}{5}$ E. $\frac{6}{11}$

29. Find the vertex of the parabola whose equation is $y = -3x^2 - 2x + 1$.

A. $\left(-\frac{1}{3}, -\frac{4}{3}\right)$ B. $\left(\frac{1}{3}, \frac{4}{3}\right)$ C. $\left(-\frac{1}{3}, \frac{4}{3}\right)$ D. $\left(-\frac{1}{3}, 1\right)$ E. None of these

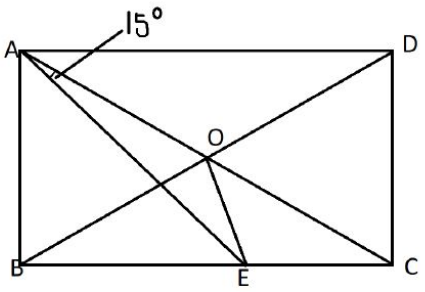
30. Suppose $\sec\theta = \dfrac{x}{y}$. Find $\cos(2\theta)$.

A. $\dfrac{2x^2-y^2}{x^2}$ B. $\dfrac{2x^2-y^2}{y^2}$ C. $\dfrac{2y^2-x^2}{x^2}$ D. $\dfrac{2y^2-x^2}{y^2}$ E. None of these

31. Consider the functions $f(x) = x^2-1, g(x) = \cos(x)$, and $h(x) = 2x+3$. Compute $h\circ(g\circ f)(x)$.

A. $2\cos(x^2-1)+3$ B. $(2x+3)\cos(x)(x^2-1)$ C. $\cos^2(2x+3)-1$ D. $2x+3\cos(x^2-1)$ E. $\sin(2x-1)+3$

32. ($\star$) ABCD is a rectangle. Point O is the intersection of two diagonals BD and AC. E is a point on BC so that AE is the bisector of $\angle BAD$. Knowing $\angle EAO$ is 15^o, what is the angle measure of BOE?



A. 60^o B. 65^o C. 70^o D. 75^o E. 80^o

33. Given that $x = 8756(a-b)$, $y = 8756(b-c)$, and $z = 8756(c-a)$, compute $\dfrac{x^2+y^2+z^2}{xy+yz+xz}$ if $xy+yz+xz \neq 0$.

A. -2 B. -1 C. 0 D. 1 E. 2

34. Let α and β be the roots of the equation,

$$x^2+px+1=0,$$

and let γ and δ be the roots of the equation,

$$x^2+qx+1=0.$$

What is the value of the expression $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$?

A. $p+q$ B. q^2-p^2 C. p^2q^2 D. $(p+q)^2$ E. pq^2+qp^2

35. If $\log_2\pi = a$ and $\log_5\pi = b$, then what is $\log_{10}\pi$?

A. ab B. a/b C. $ab/(a+b)$ D. $1/a+1/b$ E. $a+b$