# 2021 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

## Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 6, 11, 17, 26, 34

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

## Do not open your test until instructed to do so!

**Good luck!**

1. Assume that k is a real number and $x^2 - 2kx + (8k - 15) = 0$ has no real number solutions. Then

$$|k-3| + |5-k| =$$

A. 2  B. -2  C. 8-2k  D. 2k-8  E. none of these

**Solution: A.** 2

Since the equation has no real solution, $4k^2 - 4(8k - 15) < 0$ must be satisfied, i.e.,

$$k^2 - 8k + 15 = (k-5)(k-3) < 0.$$

Solve the inequality, we have $3 < k < 5$.

Therefore $|k-3| + |5-k| = (k-3) + (5-k) = 2$.

2. Given $\overline{AB}$ with endpoint $A(-4, 10)$ and midpoint $M(-1, 3)$, what are the coordinates for endpoint $B$?

A. $\left(\frac{-5}{2}, \frac{13}{2}\right)$  B. $\left(\frac{5}{2}, \frac{-13}{2}\right)$  C. $(-2, 4)$  D. $(2, -4)$  E. $(2, 4)$

**Solution: D.** $(2, -4)$

Let B be defined by $(x_b, y_b)$. For the $x-$coordinate we have $\frac{-4 + x_b}{2} = -1$. Solving for $x_b$, we have $x_b = 2$. And the $y$-coordinate is given by $\frac{10 + y_b}{2} = 3$. Solving for $y_b$ we have $y_b = -4$. So $B$'s coordinates are $(2, -4)$.

3. Consider the sequence

$$\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \frac{1}{4 \times 5}, \ldots, \frac{1}{k(k+1)}, \ldots$$

The partial sum, $S_n$, is given by which of the following?

A. $\frac{1}{n+1}$  B. $\frac{1}{n-1}$  C. $\frac{n}{n-1}$  D. $\frac{n}{n+1}$  E. $\frac{n}{n^2-1}$

**Solution: D.** $\frac{n}{n+1}$

We see the $n^{th}$ term is $a_n = \frac{1}{n(n+1)}$. Using partial fraction decomposition we have

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

Now this is a telescoping series, and

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

4. The number 12 has six distinct factors: 1, 2, 3, 4, 6, and 12. How many distinct factors does the number $12^{12}$ have?

   A. 36 B. 144 C. 288 D. 289 E. 325

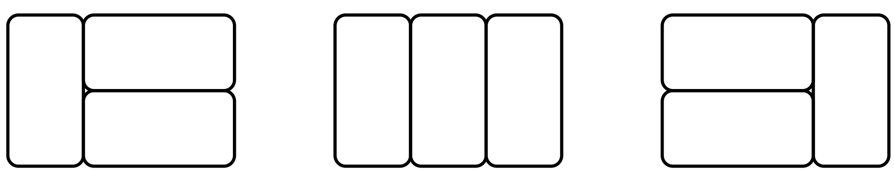
> **Solution: E.** 325
>
> The number of factors an integer has depends only on the integer's prime factorization. $12 = 2^2\,3^1$. Any factor of 12 has the form $2^a\,3^b$, where $a$ can be 0, 1, or 2 (3 possibilities), and $b$ can be 0 or 1 (two possibilities). Thus there are $3 \cdot 2 = 6$ total factors.
>
> More generally, an integer of the form $p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, where the $p_k$ are distinct primes and the $e_k$ are nonnegative integers, will have $(e_1+1)(e_2+1)\cdots(e_n+1)$ distinct factors.
>
> Note $12^{12} = (2^2\,3)^{12} = 2^{24}\,3^{12}$, so $12^{12}$ has $25 \cdot 13 = 325$ distinct factors.

5. Suppose $a$ and $b$ are integers with $a > 1$. Suppose the numbers $11b + 3$ and $55b + 52$ are both evenly divisible by $a$. Find $a$.

   A. 23 B. 37 C. 43 D. 74 E. No such $a$ exists

> **Solution: B.** 37
>
> If $11b + 3$ is evenly divisible by $a$, then $11b + 3 = na$ for some integer $n$. Likewise, $55b + 52 = ma$ for some integer $m$. Multiplying the first equation by 5 and then subtracting it from the second equation yields $37 = (m - 5n)a$. This means 37 is evenly divisible by the integers $m - 5n$ and $a$. But 37 is a prime number, so it is only divisible by 1 and itself. And since $a > 1$, we must have $a = 37$.

6. ($\star$) A $2 \times 3$ rectangle can be "tiled" by $2 \times 1$ tiles in exactly three different ways:



   In how many ways can a $2 \times 10$ rectangle be tiled using $2 \times 1$ tiles?

   A. 55 B. 76 C. 89 D. 97 E. None of these

> **Solution: C.** 89
>
> Let $T_n$ denote the number of ways to tile a $2 \times n$ rectangle. We must find $T_{10}$. We were given that $T_3 = 3$, and it's easy to see that $T_1 = 1$ and $T_2 = 2$ as well.
>
> Any tiling of a $2 \times n$ rectangle (with $n \geq 3$) must begin either with one tile placed vertically or two tiles placed horizontally:
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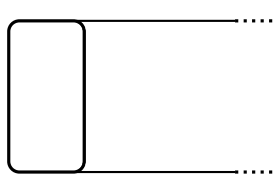
> 
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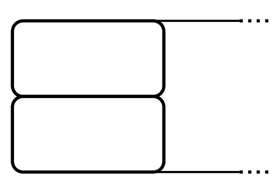
> or
>
> 

In the first case, there are $T_{n-1}$ ways to complete the tiling, and in the second case there are $T_{n-2}$ ways to do it. Thus $T_n = T_{n-1} + T_{n-2}$. And with $T_2 = 2$ and $T_3 = 3$, we can find in succession $T_4 = 5$, $T_5 = 8$, $T_6 = 13$, $T_7 = 21$, $T_8 = 34$, $T_9 = 55$, and finally $T_{10} = 89$.

7. The World Series in major league baseball is a series of games played between the National League Champion and the American League Champion.

   The two teams play a series of games until one of the teams wins for the fourth time.

   For example, the two teams in the 2020 World Series were the Los Angeles Dodgers and the Tampa Bay Rays. The Dodgers became the champions of the season by winning games #1, #3, #5 and #6 of the series.

   Suppose in the upcoming World Series the American League team is the better team.

   Assume that the American League team has a $\frac{2}{3}$ chance of winning each game played.

   Determine the probability that this series goes more than five games.

   A. $\frac{1}{3}$ B. $\frac{4}{9}$ C. $\frac{13}{27}$ D. $\frac{40}{81}$ E. $\frac{121}{243}$

**Solution: D.** $\frac{40}{81}$

We will compute the probability of the complement event, i.e. the series being won in 4 or 5 games.

The better team wins in 4 games with probability equal to $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$.

The weaker team wins in 4 games with probability equal to $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$.

Thus, the probability that the series lasts 4 games equals $\frac{17}{81}$.

The better team wins in 5 games if and only if they win exactly 3 of the first 4 games and then win game #5. There are $\binom{4}{3} = 4$ mutually exclusive ways for this to occur, each having probability equal to $\left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} = \frac{16}{243}$.

Thus, the probability that the better team wins in 5 games equals $4\left(\frac{16}{243}\right) = \frac{64}{243}$.

Similarly, the probability that the weaker team wins in 5 games equals $4\left(\frac{2}{243}\right) = \frac{8}{243}$.

Thus, the probability that the series lasts 5 games equals $\frac{72}{243}$.

Therefore, the probability that the series lasts 4 or 5 games equals $\frac{17}{81} + \frac{72}{243} = \frac{41}{81}$.

Finally, the probability that the series goes more than 5 games equals $\frac{40}{81}$.

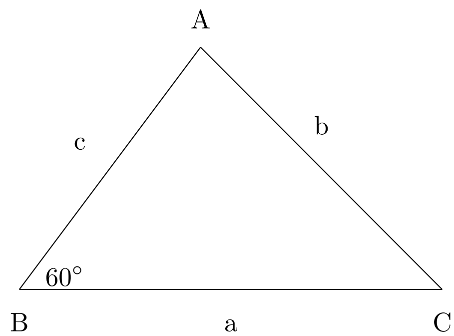
8. There are 7 bananas, 5 apples, and 3 peaches on a table. Suppose you choose 6 pieces of fruit from this stand randomly, without replacement, while blindfolded. Assuming each piece of fruit has an equal chance of being chosen, how many ways can you choose 2 bananas, 3 apples, and 1 peach?

A. 6 B. 7560 C. 3780 D. 34 E. 630

**Solution: E. 630**

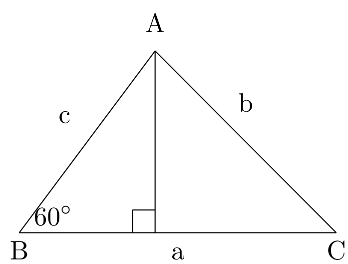
The number of ways to choose 2 bananas, 3 apples, and 1 peach from 7 bananas, 5 apples, and 3 peaches is $\binom{7}{2} \cdot \binom{5}{3} \cdot \binom{3}{1} = \frac{7!}{2!5!} \cdot \frac{5!}{2!3!} \cdot \frac{3!}{2!1!} = 7 \cdot 3 \cdot 5 \cdot 2 \cdot 3 = 630$.

9. ABC is a triangle and $\angle B = 60^o$. Let $a, b, c$ be the lengths of each side of triangle ABC as marked in the figure. What is $\frac{c}{a+b} + \frac{a}{c+b}$?




A. $\frac{1}{2}$ B. $\frac{\sqrt{2}}{2}$ C. 1 D. $\sqrt{2}$ E. 2

**Solution: C.** 1




First, drop an altitude from point $A$ onto $BC$, and let $D$ be the point of intersection. Then $DB = \frac{c}{2}$ and $AD = \frac{\sqrt{3}}{2}c$ as ABD is a right triangle.

In right triangle ADC, $DC^2 = b^2 - AD^2$. Therefore,

$(a - c/2)^2 = b^2 - \frac{3}{4}c^2$, i.e., $a^2 + c^2 = b^2 + ac$.

Notice that

$$\frac{c}{a+b} + \frac{a}{c+b} = \frac{c^2 + cb + a^2 + ab}{(a+b)(c+b)} = \frac{a^2 + c^2 + ab + bc}{ac + ab + bc + b^2} = 1.$$

10. Assume $z$ is a complex number satisfying $z(1-i) = 2i$. What is the value of z?

A. $-1-i$ B. $-1+i$ C. $1-i$ D. $1+i$ E. 2

**Solution: B.** $-1+i$

$$z = \frac{2i}{1-i} = \frac{2i(1+i)}{(1-i)(1+i)} = i(1+i) = -1+i.$$

11. ($\star$) Identical twins come from the same egg and hence are of the same sex. Fraternal twins come from separate eggs and thus have a 50-50 chance of being the same sex.

Suppose that among twins, the probability of a fraternal set equals $p$ (and thus the probability of an identical set equals $1-p$).

If a given set of twins are the same sex, what is the probability that they are identical?

A. $1-p$ B. $\frac{2(1-p)}{2-p}$ C. $\frac{3(1-p)}{3-p}$ D. $(1-p)^2$ E. None of these

**Solution: B.** $\frac{2(1-p)}{2-p}$

There are three possibilities for twins:

- A: The twins are identical and of the same sex, which has probability $1-p$.
- B: The twins are fraternal and of the same sex, which has probability $p(0.5)$.
- C: The twins are fraternal and of different sexes, which has probability $p(0.5)$.

Thus, the probability that the twins are identical given that they are the same sex is

$$\frac{P(A)}{P(A \cup B)} = \frac{1-p}{(1-p)+p(0.5)} = \frac{2(1-p)}{2-p}. \tag{1}$$

12. 10,000 high school students were selected to study whether they have had full vaccinations of COVID-19 or have already infected by COVID-19. In the survey, 6,500 students have already been vaccinated and 780 students responded that they were unfortunately infected by the virus. In this survey, 570 students said they have had the full-vaccinations and infected by COVID-19. How many of them have been fully vaccinated or been infected by COVID-19?

A. 6,710 B. 7,280 C. 6,290 D. 8,650 E. 2,720

**Solution: A.** 6,710
Let $A$ be the set of students that have been vaccinated and $B$ the set of students that have had COVID-19. Then $|A \cap B| = 570$. So the number that have **only** been vaccinated is $|A \cap B^C| = |A \setminus (A \cap B)| = 6,500 - 570 = 5,930$. And the number that have **only** gotten COVID-19 is $|B \cap A^C| = 780 - 570 = 210$. So the total number that got the vaccine or had COVID-19 is: $570 + 5,930 + 210 = 6,710$.

13. Compute the sum of all prime numbers $p$ with $100 < p < 150$.

A. 1078 B. 1090 C. 1216 D. 1346 E. None of these

**Solution: C.** 1216

Any composite number $n < 150$ must have a prime factor less than $\sqrt{150} \approx 12$. The only primes less than 12 are 2, 3, 5, 7, and 11. So begin with the numbers $\{100, 101, \ldots, 150\}$, then eliminate all multiples of 2, then all multiples of 3, then all multiples of 5, then 7, and finally 11. Any number left standing is thus prime. We find exactly ten primes between 100 and 150:

$$101, 103, 107, 109, 113, 127, 131, 137, 139, 149$$

Their sum is 1216.

14. A local restaurant allows its dinner customers to order as much as they want of the following three vegetables: green beans, spinach, and corn. A review of the 1000 dinner receipts from the previous weekend yielded the following results:

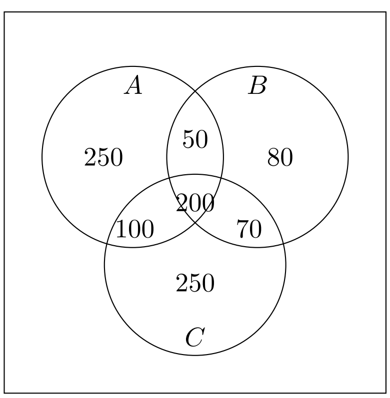
- 600 customers ordered green beans
- 400 ordered spinach
- 620 ordered corn
- 250 ordered both green beans and spinach
- 300 ordered both green beans and corn
- 270 ordered both spinach and corn
- 200 ordered all three vegetables

Given that a customer ordered green beans or corn, what is the probability that they ordered spinach?

A. less than 10% B. at least 10% but less than 20% C. at least 20% but less than 30% D. at least 30% but less than 40% E. at least 40%

**Solution: D.** at least 30% but less than 40%

Let $A$ be green beans, $B$ spinach, $C$ corn.




We want

$$P(B|A \cup C) = \frac{P(B \cap (A \cup C))}{P(A \cup C)}.$$

We see $|B \cap (A \cup C)| = 50 + 200 + 70 = 320$, so $P(B \cap (A \cup C)) = \dfrac{320}{1000} = 0.32$.

and $|(A \cup C)^C| = 80$, so $P((A \cup C)^C) = \dfrac{80}{1000} = 0.08$. So $P(A \cup C) = 1 - 0.08 = 0.92$.

Thus

$$P(B|A \cup C) = \frac{P(B \cap (A \cup C))}{P(A \cup C)} = \frac{0.32}{0.92} \approx 34.78\%.$$

15. Find the quadratic function passing though the three points: $(2, 0)$, $(1, 1)$, and $(3, 5)$.

A. $y = -3x^2 + 8x - 4$ B. $y = 0.5x^2 - 1.5x - 1$ C. $y = 3x^2 - 10x + 8$ D. $y = 3x^2 + 8x - 10$
E. None of these

**Solution: C.** $y = 3x^2 - 10x + 8$

Let $y = ax^2 + bx + c$ be the quadratic function passing through the three points.

Then plugging in each of the points into the quadratic equation yields:
$0 = 4a + 2b + c$,
$1 = a + b + c$,
$5 = 9a + 3b + c$.

We find $-1 = 3a + b$ and $5 = 5a + b$ by canceling out $c$ by means of subtracting appropriate equations from each other. We then have $a = 3$ and $b = -10$. Substituting these back into one of the equations involving $a, b$ and $c$, we get $c = 8$. Thus, $y = 3x^2 - 10x + 8$.

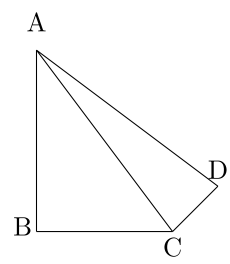
16. Which of the following is equivalent to $|-\sqrt{2} + \sqrt{3}| + |\sqrt{2} - \sqrt{3}|$?

A. $2\sqrt{2} - 2\sqrt{3}$ B. $2\sqrt{3} - 2\sqrt{2}$ C. $2\sqrt{2} + 2\sqrt{2}$ D. $2\sqrt{2} + 2\sqrt{3}$ E. None of these

**Solution: B.** $2\sqrt{3} - 2\sqrt{2}$

$|-\sqrt{2} + \sqrt{3}| + |\sqrt{2} - \sqrt{3}| = -\sqrt{2} + \sqrt{3} + -\sqrt{2} + \sqrt{3} = -2\sqrt{2} + 2\sqrt{3}$.

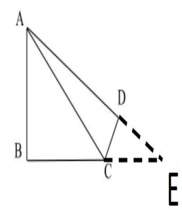
17. ($\star$) Below is a quadrilateral with $\angle ADC = \angle ABC$, $\angle BAD = 60^o$ and $\angle BCD = 120^o$. Assume $\overline{BC} = 12$ and $\overline{CD} = 3$. What is the length of the diagonal $\overline{AC}$?




A. $12\sqrt{2}$ B. $12\sqrt{3}$ C. $6\sqrt{7}$ D. $6\sqrt{5}$ E. None of these

**Solution: C.** $6\sqrt{7}$

We can find $\angle ADC = \angle ABC = 360^o - 60^o - 120^o)/2 = 90^o$. Now extend $\overline{AD}$ and $\overline{BC}$, and assume they intersect at point $E$.

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Note that $\angle AEB = \angle CED = 30^o$. Thus, $\overline{CE} = 2\overline{CD} = 6$, $\overline{BC} + \overline{CE} = 12 + 6 = 18$, and $\overline{AB} = \overline{BE}/\sqrt{3} = 18/\sqrt{3} = 6\sqrt{3}$.

As $ABC$ is a right triangle, we have $\overline{AC} = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{(6\sqrt{3})^2 + 12^2} = 6\sqrt{7}$.

18. Simplify $\sqrt{0.12}$.

A. $\frac{\sqrt{3}}{4}$ B. $\frac{\sqrt{3}}{2}$ C. $\frac{\sqrt{3}}{5}$ D. $\frac{\sqrt{3}}{7}$ E. $\frac{3\sqrt{2}}{7}$

**Solution: C.** $\frac{\sqrt{3}}{5}$

$\sqrt{0.12} = \sqrt{12/100} = \frac{\sqrt{12}}{\sqrt{100}} = 2\frac{\sqrt{3}}{10} = \frac{\sqrt{3}}{5}$.

19. What is the largest possible $k$ so that $y = x^2 + |2x - 4| \geq k$ always holds true?

A. $-4$ B. $-5$ C. 1 D. 3 E. 4

**Solution: D.** 3

When $x \geq 2$, $f(x) = x^2 + |2x - 4| = x^2 + 2x - 4 = (x + 1)^2 - 5$. So the minimum is when $f(2) = 4$.
When $x < 2$, $f(x) = x^2 + |2x - 4| = x^2 - 2x + 4 = (x - 1)^2 + 3$. The minimum is when $f(1) = 3$.
So the value for $k$ would be $k = 3$.

20. Find the domain of the function given by $f(x) = \frac{\sqrt{x-1}}{x^2 - 9}$.

A. $\{x \in \mathbb{R} : x \geq 1\}$ B. $\{x \in \mathbb{R} : x \geq 1 \text{ and } x \neq 3\}$ C. $\{x \in \mathbb{R} : x > 1 \text{ and } x \neq 3\}$
D. $\{x \in \mathbb{R} : x > 1 \text{ and } x \neq -3 \text{ and } x \neq 3\}$ E. $\{x \in \mathbb{R} : x \neq -3 \text{ and } x \neq 3\}$

**Solution: B.** $\{x \in \mathbb{R} : x \geq 1 \text{ and } x \neq 3\}$

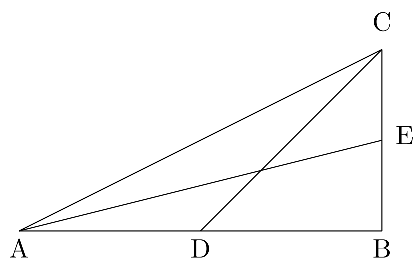
From the denominator we see that $x^2 \neq 9$ so $x \neq -3$ or $x \neq 3$. From the numerator we need $x - 1 \geq 0$. So $x \geq 1$.

21. Choose one of the following statements that correctly describes the equations of two lines: $8x-4y+37 = 0$ and $x + 2y = 26$.

A. The lines are perpendicular B. The lines are parallel and never intersect C. The lines are identical D. The lines intersect exactly at two distinct points E. None of these

**Solution: A.** The lines are perpendicular
Solving the equation $8x - 4y + 37 = 0$ for $y$ we get $y = 2x + \frac{37}{4}$. Solving for $y$ in the equation $x + 2y = 26$ yields $y = -\frac{1}{2}x + 13$. Since the slope in the first equation is 2 and the slope in the second equation is the opposite reciprocal $-\frac{1}{2}$, these are equations of perpendicular lines.

22. Let ABC be a right triangle with $\angle ABC = 90°$. Points $D$ and $E$ are the midpoints of $AB$ and $BC$ respectively. If $\overline{CD} = 5$ and $\overline{AE} = 4\sqrt{5}$, what is the length of $AC$?




A. $6\sqrt{3}$ B. $4\sqrt{7}$ C. 10 D. $2\sqrt{21}$ E. $6\sqrt{7}$

**Solution: D.** $2\sqrt{21}$

Since $\triangle ABE$ is a right triangle, $BE^2 + AB^2 = AE^2$. So $\left(\frac{1}{2}BC\right)^2 + AB^2 = AE^2$.

Likewise, for $\triangle BCD$ we have $BC^2 + \left(\frac{1}{2}AB\right)^2 = CD^2$.

Adding these equations, we have $\frac{5}{4}(BC^2 + AB^2) = AE^2 + CD^2 = 5^2 + (4\sqrt{5})^2 = 105$.

Thus, $AC = \sqrt{BC^2 + AB^2} = \sqrt{105 \cdot \frac{4}{5}} = 2\sqrt{21}$.

23. Let $x$ and $y$ be natural numbers such that $x = a^m b^n$ and $y = a^r b^s$ where $a, b, m, n, r, s$ are all natural numbers. Find $\gcd(x, y)$ if $\gcd(a, b) = 1$.

A. $a^{\min(m,r)}b^{\min(n,s)}$ B. $a^{\min(n,r)}b^{\min(m,s)}$ C. $a^{\max(m,n)}b^{\max(r,s)}$ D. $a^{\max(m,r)}b^{\max(n,s)}$
E. None of these

**Solution: A.** $a^{\min(m,r)}b^{\min(n,s)}$
Let $G$ denote $\gcd(x,y)$. Since $\gcd(a,b)=1$, $G$ consists of as many powers of $a$ and $b$ that appear in both $x$ and $y$. Thus, the highest power of $a$ that can appear as a factor of $G$ is $\min(m,r)$. Similarly, the highest power of $b$ that can be a factor if $G$ is $\min(n,s)$. Hence, $G = a^{\min(m,r)}b^{\min(n,s)}$.

24. Suppose you have 200 fish in an aquarium tank, and 99% of them are guppies. If you want to reduce the proportion of guppies in the tank to exactly 98%, how many guppies would you have to remove?

A. 1 B. 2 C. 20 D. 50 E. 100

**Solution:**
**E.** 100

There are originally 198 guppies in the tank. Suppose you remove $x$ guppies. To have 98% of the tank guppies, we just need to solve the equation

$$\frac{198-x}{200-x} = 0.98$$

for $x$. Doing so, we see $x = 100$.

25. Solve the system of inequalities:

$$x + 6 \leq 4x,$$
$$1 - 3x < 15 - 5x.$$

A. $(2,7)$ B. $(-\infty,2)\cup[7,\infty)$ C. $[2,7]$ D. $(2,7]$ E. $[2,7)$

**Solution: E.** $[2,7)$
Starting with $x+6 \leq 4x$, we get $6 \leq 3x$. Thus, $x \geq 2$.
From $1-3x < 15-5x$, we obtain $2x < 14$, and so $x < 7$.
Hence, the values that satisfy both inequalities are $[2,7)$.

26. ($\star$) What is the sum of the coefficients of the polynomial obtained after expanding and collecting the terms of the product $(1-5x+5x^2)^{2020}(1+5x-5x^2)^{2021}$?

A. 1 B. $1+5^{2020}$ C. $1-5^{2020}+5^{2021}$ D. $1+5^{2020}-5^{2021}$ E. None of these.

**Solution:**
**A.** 1

The expansion of the given expression will have the form,

$$(1 - 5x + 5x^2)^{2020}(1 + 5x - 5x^2)^{2021} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where the order of the resulting polynomial is $n = 2(2020) + 2(2021)$. If we let $x = 1$ in the above expression, we have

$$1^{2020} \cdot 1^{2021} = a_0 + a_1 + a_2 + \cdots + a_n,$$

and thus the sum we seek is 1.

27. Consider the repeating decimal given by $0.\overline{5} = 0.555\ldots$. Express this decimal in terms of a single fraction.
A. $\frac{5}{9}$ B. $\frac{55}{90}$ C. $\frac{56}{100}$ D. $\frac{55}{100}$ E. $\frac{556}{999}$

**Solution: A. $\frac{5}{9}$**
Let $x = 0.555\ldots$.
Then, $10x = 5.555\ldots$.
Thus, $10xx = 5$ to obtain $9x = 5$.
Therefore, $x = \frac{5}{9}$.

28. Let $f(x)$ and $g(x)$ denote two distinct linear functions in the $xy$-plane. Note that "distinct" means that the two functions are not equal. Choose all possible correct statements.
(a) There are $f(x)$ and $g(x)$ which do not intersect.
(b) There are $f(x)$ and $g(x)$ which intersect at exactly one point.
(c) There are $f(x)$ and $g(x)$ which intersect at exactly two points.
(d) There are $f(x)$ and $g(x)$ which intersect at exactly three points.
(e) There are $f(x)$ and $g(x)$ which intersect at infinitely many points.

A. Only statement (a) can be true.
B. Only statement (b) can be true.
C. Only statement (e) can be true.
D. Only statements (b), (c), and (d) can be true.
E. Only statements (a) and (b) can be true.

**Solution: E. Only statements (a) and (b) can be true.**

Statements (c) and (d) cannot be correct since two lines either intersect at exactly one or infinitely many points if they intersect at all. Statement (e) cannot be true since the two functions meet at infinitely many points, which means $f(x)$ and $g(x)$ are identical functions. Our assumption is that the functions are distinct and therefore (a) and (b) are the two possible true statements under the given assumptions.

29. From time to time Mary will write a letter to her mother Ann. Whenever Ann receives a letter from Mary, she always sends Mary a letter in reply. Mary writes a letter to her mother and does not receive a letter in reply.

Assuming that each sent letter has a one in $n$ chance of getting lost in the mail, find the probability that Ann received Mary's letter.

A. $\dfrac{n-1}{n+1}$ B. $\dfrac{n^2-1}{n^2+1}$ C. $\dfrac{n-1}{2n-1}$ D. $\dfrac{n^2-1}{2n^2-1}$ E. $\dfrac{n^2-1}{2n^2+1}$

**Solution: C.** $\dfrac{n-1}{2n-1}$

Mary does not receive a reply in one of two cases.

**Case 1:** Ann receives Mary's letter, and Ann's reply is lost. The probability of this happening is $\left(\dfrac{n-1}{n}\right)\dfrac{1}{n}=\dfrac{n-1}{n^2}$.

**Case 2:** Ann didn't receive the letter. The probability of this happening is $\dfrac{1}{n}$

We want the probability that Ann received the letter but Mary didn't get a reply. Using conditional probability,

$$P(\text{Ann received}|\text{Mary didn't get a reply})=\frac{P(\text{Ann received}\cap\text{Mary didn't get a reply})}{P(\text{Mary didn't get a reply})}$$

Now Case 1 above tells us:

$$P(\text{Ann received}\cap\text{Mary didn't get a reply})=\frac{n-1}{n^2}$$

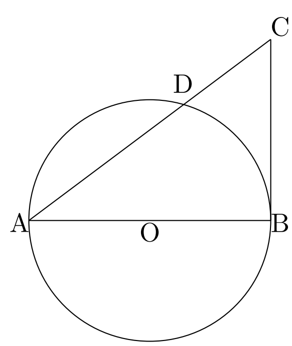
And

$$P(\text{Mary didn't get a reply})=\frac{n-1}{n^2}+\frac{1}{n}=\frac{2n-1}{n^2}$$

Thus

$$\frac{P(\text{Ann received}\cap\text{Mary didn't get a reply})}{P(\text{Mary didn't get a reply})}=\frac{(n-1)/n^2}{(2n-1)/n^2}=\frac{n-1}{2n-1}$$

30. Triangle $ABC$ is a right triangle with $\angle ABC=90^o$. Point $D$ is the intersection of $AC$ and the circle whose diameter is $AB$. If $\overline{CD}=6$, $\overline{BC}=20$, what is the radius of the circle?




A. $6\sqrt{78}$ B. 27 C. $5\sqrt{13}$ D. $\dfrac{10}{3}\sqrt{91}$ E. $\sqrt{69}$

**Solution: D.** $\frac{10}{3}\sqrt{91}$

As triangle $ABC$ and triangle $BDC$ are similar, $\frac{\overline{AC}}{\overline{BC}} = \frac{\overline{BC}}{\overline{CD}}$.

Therefore, $\overline{AC} = \frac{20^2}{6} = \frac{200}{3}$.

Let $r$ denote the radius of the circle. In triangle $ABC$, $4r^2 + 20^2 = \overline{AC}^2$. So $4r^2 + 20^2 = (200/3)^2$.

Thus $r = \frac{10}{3}\sqrt{91}$.

31. Let $x$ be a positive real number satisfying $\sqrt{2x^2 + 5x + 13} + \sqrt{2x^2 + 5x - 2} = 5$. What is the value of $x$?

A. $\frac{1}{2}$ B. $\frac{2}{5}$ C. $\frac{1}{5}$ D. $\frac{5}{11}$ E. None of these

**Solution: A.** $\frac{1}{2}$

Let $y = 2x^2 + 5x + 13$. Then $2x^2 + 5x - 2 = 2x^2 + 5x + 13 - 15 = y - 15$.

So we have

$$\sqrt{y} + \sqrt{y - 15} = 5.$$

That is $\sqrt{y - 15} = 5 - \sqrt{y}$. Squaring both sides we have

$$y - 15 = 25 - 10\sqrt{y} + y.$$

Simplifying, we have $40 = 10\sqrt{y}$, so that $\sqrt{y} = 4$ and $y = 16$.

Then $2x^2 + 5x + 13 = 16$, or $2x^2 + 5x - 3 = 0$. So $(2x - 1)(x + 3) = 0$. Thus our solutions are $x = \frac{1}{2}, x = -3$.

32. Let $f(x) = \frac{4x - 3}{2x + 1}$. Find the inverse function $f^{-1}(x)$.

A. $f^{-1}(x) = \frac{2x - 4}{-x - 3}$ B. $f^{-1}(x) = \frac{2x + 4}{x + 3}$ C. $f^{-1}(x) = \frac{-x - 3}{2x - 4}$ D. $f^{-1}(x) = \frac{x + 3}{2x - 4}$
E. $f^{-1}(x) = \frac{-x + 3}{-2x + 4}$

**Solution: C.** $f^{-1}(x) = \frac{-x - 3}{2x - 4}$

Let $y = \frac{4x - 3}{2x + 1}$. We have

$$y(2x + 1) = 4x - 3.$$

So

$$2xy + y = 4x - 3.$$

Thus

$$x(2y - 4) = -y - 3.$$

Therefore

$$x = \frac{-y-3}{2y-4}$$

Switching variables we see the inverse is $y = \dfrac{-x-3}{2x-4}$.

33. Two players put a dollar into a pot. They decide to throw a pair of unbalanced (i.e. unfair) dice alternately. The first player who throws a sum of seven wins the pot.

    Suppose the player who goes first has probability equal to $\frac{5}{8}$ of eventually winning the game.

    Let $p$ denote the probability of rolling a sum of seven with this pair of dice.

    Which one of the following statements is true concerning the value of $p$?

    A. $0.16 < p < 0.30$  B. $0.31 < p < 0.45$  C. $0.46 < p < 0.60$  D. $0.61 < p < 0.75$
    E. $0.76 < p < 0.90$

**Solution: B.** $0.31 < p < 0.45$

There are an infinite number of ways in which the 1st player wins.

$S, FFS, FFFFS, FFFFFFS, \ldots$

where $S$ denotes winning on their 1st turn, $FFS$ denotes winning on their 2nd turn, $FFFFS$ denotes winning on their 3rd turn, and so on.

Letting $p$ denote the probability of rolling a sum of seven, we see the following:

$P(S) = p$
$P(FFS) = (1-p)^2 p$
$P(FFFFS) = (1-p)^4 p$
$\cdots$

Thus, player #1s probability of winning is an infinite geometric series with 1st term equal to $p$ and common ratio equal to $(1-p)^2$ (this ratio is between 0 and 1).

So P(Player #1 wins)=$\dfrac{p}{1-(1-p)^2}$.

Setting this equal to $\frac{5}{8}$ and solving for $p$ yields $p = \frac{2}{5} = 0.40$.

34. ($\star$) There are three boxes of donuts. Each box contains donuts filled with exactly one type of filling, either raspberry or chocolate, as follows:

    - Box #1 contains $r$ raspberry and $c$ chocolate filled donuts.
    - Box #2 contains 1 raspberry and 2 chocolate filled donuts.
    - Box #3 contains 3 raspberry and 2 chocolate filled donuts.

    A donut is randomly selected from Box #1 and placed in Box #2.

    Then, a donut is randomly selected from Box #2 and placed in Box #3.

    Finally, a donut is randomly selected from Box #3.

    The probability that a raspberry-filled donut was selected from Box #3 is $\frac{11}{20}$.

    What proportion of the original $r + c$ donuts in Box #1 were raspberry-filled?

A. $\frac{1}{6}$ B. $\frac{1}{5}$ C. $\frac{1}{4}$ D. $\frac{3}{8}$ E. None of these

**Solution: B.** $\frac{1}{5}$

Let $R_1R_2R_3$ denote the outcome of choosing a raspberry-filled donut on the first draw, the second draw, and the third draw.

Then $P(R_1 \cap R_2 \cap R_3) = P(R_1) \cdot P(R_2|R_1) \cdot P(R_3|(R_1 \cap R_2)) = p \cdot \frac{2}{4} \cdot \frac{4}{6} = \frac{p}{3}$, where $p$ denotes the original proportion of raspberry-filled donuts in Box #1

There are 4 distinct ways to select a raspberry donut from Box #3, as follows:

- $R_1R_2R_3$ with probability $p \cdot \frac{2}{4} \cdot \frac{4}{6} = \frac{p}{3}$
- $R_1C_2R_3$ with probability $p \cdot \frac{2}{4} \cdot \frac{3}{6} = \frac{p}{4}$
- $C_1R_2R_3$ with probability $(1-p) \cdot \frac{1}{4} \cdot \frac{4}{6} = \frac{1-p}{6}$
- $C_1C_2R_3$ with probability $(1-p) \cdot \frac{3}{4} \cdot \frac{3}{6} = \frac{3(1-p)}{8}$

Adding the probabilities, we find that the probability of choosing a raspberry-filled donut from Box #3 is $\frac{13+p}{24}$. Setting this equal to $\frac{11}{20}$ and solving for $p$, yields $p = \frac{1}{5}$.

35. Calculate $\frac{1}{2+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + \cdots + \frac{1}{2019\sqrt{2018}+2018\sqrt{2019}} + \frac{1}{2020\sqrt{2019}+2019\sqrt{2020}}$.

A. $\frac{\sqrt{2019}}{\sqrt{2020}}$ B. $\frac{\sqrt{2020}}{\sqrt{2019}}$ C. $1 - \frac{1}{\sqrt{2020}}$ D. $1 + \frac{1}{\sqrt{2020}} - \frac{1}{\sqrt{2019}}$ E. $1 - \frac{1}{\sqrt{2019}}$

**Solution: C.** $1 - \frac{1}{\sqrt{2020}}$

Note that $\frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$

$= \frac{1}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})}$

$= \frac{1}{\sqrt{n(n+1)}} \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{1}{\sqrt{n(n+1)}} \cdot \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n}$

$= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}$

$= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$.

Therefore, $\frac{1}{2+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + \cdots$

$+ \frac{1}{2019\sqrt{2018}+2018\sqrt{2019}} + \frac{1}{2020\sqrt{2019}+2019\sqrt{2020}}$

$$= \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) + \cdots + \left(\frac{1}{\sqrt{2018}} - \frac{1}{\sqrt{2019}}\right) + \left(\frac{1}{\sqrt{2019}} - \frac{1}{\sqrt{2020}}\right)$$
$$= 1 - \frac{1}{\sqrt{2020}}.$$