2021 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 6, 11, 17, 26, 34

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Assume that k is a real number and $x^2 - 2kx + (8k - 15) = 0$ has no real number solutions. Then
$$|k-3| + |5-k| =$$
A. 2 B. -2 C. 8-2k D. 2k-8 E. none of these

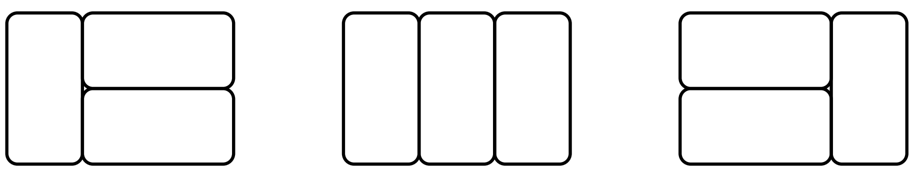
2. Given $\overline{AB}$ with endpoint $A(-4, 10)$ and midpoint $M(-1, 3)$, what are the coordinates for endpoint B?
A. $\left(\frac{-5}{2}, \frac{13}{2}\right)$ B. $\left(\frac{5}{2}, \frac{-13}{2}\right)$ C. $(-2, 4)$ D. $(2, -4)$ E. $(2, 4)$

3. Consider the sequence
$$\frac{1}{1\times 2}, \frac{1}{2\times 3}, \frac{1}{3\times 4}, \frac{1}{4\times 5}, \ldots, \frac{1}{k(k+1)}, \ldots$$
The partial sum, S_n, is given by which of the following?
A. $\frac{1}{n+1}$ B. $\frac{1}{n-1}$ C. $\frac{n}{n-1}$ D. $\frac{n}{n+1}$ E. $\frac{n}{n^2-1}$

4. The number 12 has six distinct factors: 1, 2, 3, 4, 6, and 12. How many distinct factors does the number 12^{12} have?
A. 36 B. 144 C. 288 D. 289 E. 325

5. Suppose a and b are integers with $a > 1$. Suppose the numbers $11b + 3$ and $55b + 52$ are both evenly divisible by a. Find a.
A. 23 B. 37 C. 43 D. 74 E. No such a exists

6. ($\star$) A 2×3 rectangle can be "tiled" by 2×1 tiles in exactly three different ways:



In how many ways can a 2×10 rectangle be tiled using 2×1 tiles?
A. 55 B. 76 C. 89 D. 97 E. None of these

7. The World Series in major league baseball is a series of games played between the National League Champion and the American League Champion.

The two teams play a series of games until one of the teams wins for the fourth time.

For example, the two teams in the 2020 World Series were the Los Angeles Dodgers and the Tampa Bay Rays. The Dodgers became the champions of the season by winning games #1, #3, #5 and #6 of the series.

Suppose in the upcoming World Series the American League team is the better team.

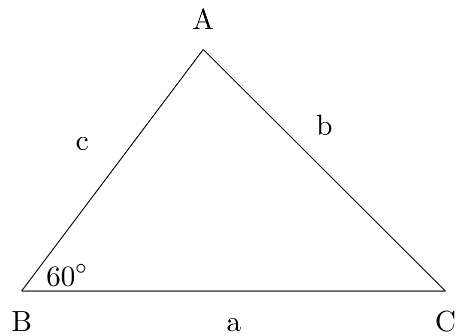
Assume that the American League team has a $\frac{2}{3}$ chance of winning each game played.

Determine the probability that this series goes more than five games.
A. $\frac{1}{3}$ B. $\frac{4}{9}$ C. $\frac{13}{27}$ D. $\frac{40}{81}$ E. $\frac{121}{243}$

8. There are 7 bananas, 5 apples, and 3 peaches on a table. Suppose you choose 6 pieces of fruit from this stand randomly, without replacement, while blindfolded. Assuming each piece of fruit has an equal chance of being chosen, how many ways can you choose 2 bananas, 3 apples, and 1 peach?

   A. 6 B. 7560 C. 3780 D. 34 E. 630

9. ABC is a triangle and $\angle B = 60^o$. Let a, b, c be the lengths of each side of triangle ABC as marked in the figure. What is $\dfrac{c}{a+b} + \dfrac{a}{c+b}$?



   A. $\dfrac{1}{2}$ B. $\dfrac{\sqrt{2}}{2}$ C. 1 D. $\sqrt{2}$ E. 2

10. Assume z is a complex number satisfying $z(1-i) = 2i$. What is the value of z?

    A. $-1-i$ B. $-1+i$ C. $1-i$ D. $1+i$ E. 2

11. ($\star$) Identical twins come from the same egg and hence are of the same sex. Fraternal twins come from separate eggs and thus have a 50-50 chance of being the same sex.

    Suppose that among twins, the probability of a fraternal set equals p (and thus the probability of an identical set equals $1-p$).

    If a given set of twins are the same sex, what is the probability that they are identical?

    A. $1-p$ B. $\dfrac{2(1-p)}{2-p}$ C. $\dfrac{3(1-p)}{3-p}$ D. $(1-p)^2$ E. None of these

12. 10,000 high school students were selected to study whether they have had full vaccinations of COVID-19 or have already infected by COVID-19. In the survey, 6,500 students have already been vaccinated and 780 students responded that they were unfortunately infected by the virus. In this survey, 570 students said they have had the full-vaccinations and infected by COVID-19. How many of them have been fully vaccinated or been infected by COVID-19?

    A. 6,710 B. 7,280 C. 6,290 D. 8,650 E. 2,720

13. Compute the sum of all prime numbers p with $100 < p < 150$.

    A. 1078 B. 1090 C. 1216 D. 1346 E. None of these

14. A local restaurant allows its dinner customers to order as much as they want of the following three vegetables: green beans, spinach, and corn. A review of the 1000 dinner receipts from the previous weekend yielded the following results:

- 600 customers ordered green beans
- 400 ordered spinach
- 620 ordered corn
- 250 ordered both green beans and spinach
- 300 ordered both green beans and corn
- 270 ordered both spinach and corn
- 200 ordered all three vegetables

Given that a customer ordered green beans or corn, what is the probability that they ordered spinach?

A. less than 10% B. at least 10% but less than 20% C. at least 20% but less than 30% D. at least 30% but less than 40% E. at least 40%

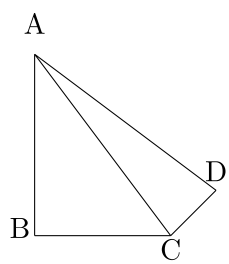
15. Find the quadratic function passing though the three points: $(2, 0)$, $(1, 1)$, and $(3, 5)$.

A. $y = -3x^2 + 8x - 4$ B. $y = 0.5x^2 - 1.5x - 1$ C. $y = 3x^2 - 10x + 8$ D. $y = 3x^2 + 8x - 10$ E. None of these

16. Which of the following is equivalent to $|-\sqrt{2} + \sqrt{3}| + |\sqrt{2} - \sqrt{3}|$?

A. $2\sqrt{2} - 2\sqrt{3}$ B. $2\sqrt{3} - 2\sqrt{2}$ C. $2\sqrt{2} + 2\sqrt{2}$ D. $2\sqrt{2} + 2\sqrt{3}$ E. None of these

17. ($\star$) Below is a quadrilateral with $\angle ADC = \angle ABC$, $\angle BAD = 60^o$ and $\angle BCD = 120^o$. Assume $\overline{BC} = 12$ and $\overline{CD} = 3$. What is the length of the diagonal $\overline{AC}$?



A. $12\sqrt{2}$ B. $12\sqrt{3}$ C. $6\sqrt{7}$ D. $6\sqrt{5}$ E. None of these

18. Simplify $\sqrt{0.12}$.

A. $\dfrac{\sqrt{3}}{4}$ B. $\dfrac{\sqrt{3}}{2}$ C. $\dfrac{\sqrt{3}}{5}$ D. $\dfrac{\sqrt{3}}{7}$ E. $\dfrac{3\sqrt{2}}{7}$

19. What is the largest possible k so that $y = x^2 + |2x - 4| \geq k$ always holds true?

A. -4 B. -5 C. 1 D. 3 E. 4

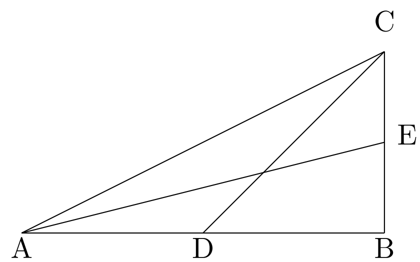
20. Find the domain of the function given by $f(x) = \dfrac{\sqrt{x-1}}{x^2-9}$.

    A. $\{x \in \mathbb{R} : x \geq 1\}$ B. $\{x \in \mathbb{R} : x \geq 1 \text{ and } x \neq 3\}$ C. $\{x \in \mathbb{R} : x > 1 \text{ and } x \neq 3\}$ D. $\{x \in \mathbb{R} : x > 1 \text{ and } x \neq -3 \text{ and } x \neq 3\}$ E. $\{x \in \mathbb{R} : x \neq -3 \text{ and } x \neq 3\}$

21. Choose one of the following statements that correctly describes the equations of two lines: $8x - 4y + 37 = 0$ and $x + 2y = 26$.

    A. The lines are perpendicular B. The lines are parallel and never intersect C. The lines are identical D. The lines intersect exactly at two distinct points E. None of these

22. Let ABC be a right triangle with $\angle ABC = 90°$. Points D and E are the midpoints of AB and BC respectively. If $\overline{CD} = 5$ and $\overline{AE} = 4\sqrt{5}$, what is the length of AC?



    A. $6\sqrt{3}$ B. $4\sqrt{7}$ C. 10 D. $2\sqrt{21}$ E. $6\sqrt{7}$

23. Let x and y be natural numbers such that $x = a^m b^n$ and $y = a^r b^s$ where a, b, m, n, r, s are all natural numbers. Find $\gcd(x, y)$ if $\gcd(a, b) = 1$.

    A. $a^{\min(m,r)} b^{\min(n,s)}$ B. $a^{\min(n,r)} b^{\min(m,s)}$ C. $a^{\max(m,n)} b^{\max(r,s)}$ D. $a^{\max(m,r)} b^{\max(n,s)}$ E. None of these

24. Suppose you have 200 fish in an aquarium tank, and 99% of them are guppies. If you want to reduce the proportion of guppies in the tank to exactly 98%, how many guppies would you have to remove?

    A. 1 B. 2 C. 20 D. 50 E. 100

25. Solve the system of inequalities:

$$x + 6 \leq 4x,$$
$$1 - 3x < 15 - 5x.$$

    A. $(2, 7)$ B. $(-\infty, 2) \cup [7, \infty)$ C. $[2, 7]$ D. $(2, 7]$ E. $[2, 7)$

26. ($\star$) What is the sum of the coefficients of the polynomial obtained after expanding and collecting the terms of the product $(1 - 5x + 5x^2)^{2020}(1 + 5x - 5x^2)^{2021}$?

    A. 1 B. $1 + 5^{2020}$ C. $1 - 5^{2020} + 5^{2021}$ D. $1 + 5^{2020} - 5^{2021}$ E. None of these.

27. Consider the repeating decimal given by $0.\overline{5} = 0.555\ldots$. Express this decimal in terms of a single fraction.

    A. $\dfrac{5}{9}$ B. $\dfrac{55}{90}$ C. $\dfrac{56}{100}$ D. $\dfrac{55}{100}$ E. $\dfrac{556}{999}$

28. Let $f(x)$ and $g(x)$ denote two distinct linear functions in the xy-plane. Note that "distinct" means that the two functions are not equal. Choose all possible correct statements.

(a) There are $f(x)$ and $g(x)$ which do not intersect.
(b) There are $f(x)$ and $g(x)$ which intersect at exactly one point.
(c) There are $f(x)$ and $g(x)$ which intersect at exactly two points.
(d) There are $f(x)$ and $g(x)$ which intersect at exactly three points.
(e) There are $f(x)$ and $g(x)$ which intersect at infinitely many points.

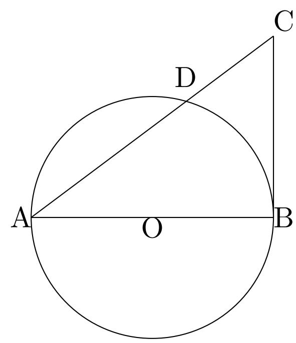
A. Only statement (a) can be true.
B. Only statement (b) can be true.
C. Only statement (e) can be true.
D. Only statements (b), (c), and (d) can be true.
E. Only statements (a) and (b) can be true.

29. From time to time Mary will write a letter to her mother Ann. Whenever Ann receives a letter from Mary, she always sends Mary a letter in reply. Mary writes a letter to her mother and does not receive a letter in reply.

Assuming that each sent letter has a one in n chance of getting lost in the mail, find the probability that Ann received Mary's letter.

A. $\dfrac{n-1}{n+1}$ B. $\dfrac{n^2-1}{n^2+1}$ C. $\dfrac{n-1}{2n-1}$ D. $\dfrac{n^2-1}{2n^2-1}$ E. $\dfrac{n^2-1}{2n^2+1}$

30. Triangle ABC is a right triangle with $\angle ABC = 90^o$. Point D is the intersection of AC and the circle whose diameter is AB. If $\overline{CD} = 6$, $\overline{BC} = 20$, what is the radius of the circle?



A. $6\sqrt{78}$ B. 27 C. $5\sqrt{13}$ D. $\dfrac{10}{3}\sqrt{91}$ E. $\sqrt{69}$

31. Let x be a positive real number satisfying $\sqrt{2x^2+5x+13}+\sqrt{2x^2+5x-2}=5$. What is the value of x?

A. $\dfrac{1}{2}$ B. $\dfrac{2}{5}$ C. $\dfrac{1}{5}$ D. $\dfrac{5}{11}$ E. None of these

32. Let $f(x) = \dfrac{4x-3}{2x+1}$. Find the inverse function $f^{-1}(x)$.

A. $f^{-1}(x) = \dfrac{2x-4}{-x-3}$ B. $f^{-1}(x) = \dfrac{2x+4}{x+3}$ C. $f^{-1}(x) = \dfrac{-x-3}{2x-4}$ D. $f^{-1}(x) = \dfrac{x+3}{2x-4}$
E. $f^{-1}(x) = \dfrac{-x+3}{-2x+4}$

33. Two players put a dollar into a pot. They decide to throw a pair of unbalanced (i.e. unfair) dice alternately. The first player who throws a sum of seven wins the pot.

    Suppose the player who goes first has probability equal to $\frac{5}{8}$ of eventually winning the game.

    Let p denote the probability of rolling a sum of seven with this pair of dice.

    Which one of the following statements is true concerning the value of p?

    A. $0.16 < p < 0.30$ B. $0.31 < p < 0.45$ C. $0.46 < p < 0.60$ D. $0.61 < p < 0.75$ E. $0.76 < p < 0.90$

34. ($\star$) There are three boxes of donuts. Each box contains donuts filled with exactly one type of filling, either raspberry or chocolate, as follows:

    - Box #1 contains r raspberry and c chocolate filled donuts.
    - Box #2 contains 1 raspberry and 2 chocolate filled donuts.
    - Box #3 contains 3 raspberry and 2 chocolate filled donuts.

    A donut is randomly selected from Box #1 and placed in Box #2.

    Then, a donut is randomly selected from Box #2 and placed in Box #3.

    Finally, a donut is randomly selected from Box #3.

    The probability that a raspberry-filled donut was selected from Box #3 is $\frac{11}{20}$.

    What proportion of the original $r + c$ donuts in Box #1 were raspberry-filled?

    A. $\frac{1}{6}$ B. $\frac{1}{5}$ C. $\frac{1}{4}$ D. $\frac{3}{8}$ E. None of these

35. Calculate $\frac{1}{2+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + \cdots + \frac{1}{2019\sqrt{2018}+2018\sqrt{2019}} + \frac{1}{2020\sqrt{2019}+2019\sqrt{2020}}$.

    A. $\frac{\sqrt{2019}}{\sqrt{2020}}$ B. $\frac{\sqrt{2020}}{\sqrt{2019}}$ C. $1 - \frac{1}{\sqrt{2020}}$ D. $1 + \frac{1}{\sqrt{2020}} - \frac{1}{\sqrt{2019}}$ E. $1 - \frac{1}{\sqrt{2019}}$