2019 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 5, 11, 14 , 20, 28

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. A square has an area of 49 square inches. If the same amount is added to the length and removed from the width, the resulting rectangle has an area of 45 square inches. Find the dimensions of the rectangle in inches.

   A. 5×9 B. 4×9 C. 3×4 D. 5×10 E. 5×6

   Solution: A. 5×9

   Since the Area of the square is 49, we know that each side length is 7. For the rectangle we will assign sides $7 - x$ and $x + 7$. Then $(7 - x)(7 + x) = 45$, which gives $x^2 = 4$, and so $x = 2$. Then the side lengths of the rectangle are $7 - 2 = 5$ and $2 + 7 = 9$. So the rectangle is 5 in by 9 in.

2. What is the smallest positive integer that is divisible by the first 12 composite numbers?

   A. 840 B. 10080 C. 2520 D. 5040 E. 27720

   Solution: D. 5040

   We are looking for the least common multiple of 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, and 21. Factoring each of these numbers, we find that the LCM is $2^4 \cdot 3^2 \cdot 5 \cdot 7 = 5040$.

3. A boy who runs 8 miles per hour is 3/8 of the way through a railroad tunnel when he hears a train whistle behind him. If he runs back, he will leave the tunnel precisely the moment the train enters it. If instead he keeps on running to the far end of the tunnel, the train will reach him just as he leaves the tunnel.

   What is the speed of the train in miles per hour?

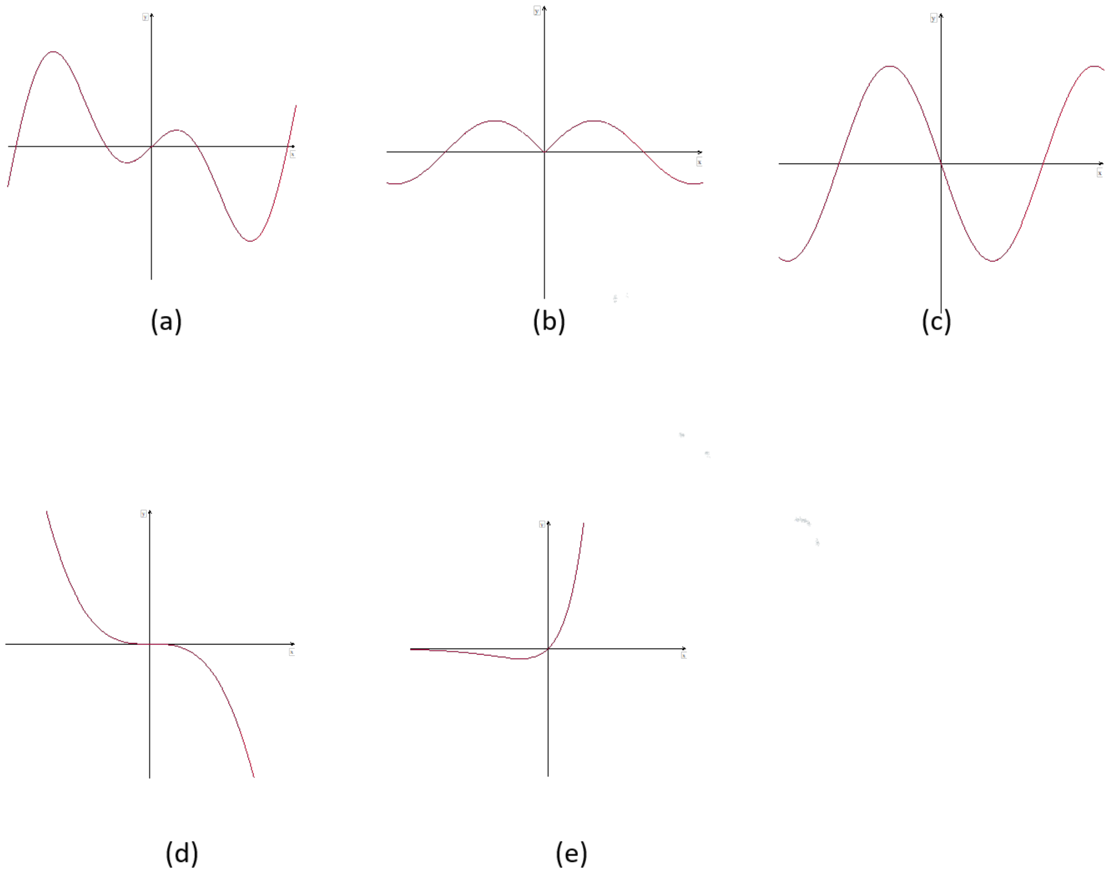
   A. 24 B. 28 C. 32 D. 36 E. 40

   Solution: C. 32

   Suppose the boy runs towards the far end of the tunnel. He is 3/4 of the way to the end of the tunnel $\left(\frac{3}{8} + \frac{3}{8}\right)$ when the train enters the tunnel.

   In order to catch the boy just as the boy leaves the tunnel, the train must travel 4 times as fast as the boy. So the speed of the train is 32.

4. Assume f(x) is an odd function. Which of the following is the graph of $y = -xf(x)$?



Solution: B. Note that both $y = f(x)$ and $y = -x$ are odd functions. Thus, $g(x) = -xf(x)$ is an even function, since $g(-x) = -(-x)f(-x) = x(-f(x)) = -xf(x) = g(x)$. Graph B is the only graph of an even function, with symmetry about the y-axis, and therefore must be the answer.

5. $\star$ How many nonnegative integers less than 2019 are not solutions to $x^7 + 2x^4 - x^3 + 3 \equiv 0 \mod 5$?

A. 402 B. 403 C. 404 D. 405 E. None of These

Solution: C: 404

We have

$$\begin{aligned} x^7 + 2x^4 - x^3 + 3 &\equiv x^7 - x^3 + 2x^4 - 2 \\ &\equiv x^3(x^4 - 1) + 2(x^4 - 1) \\ &\equiv (x^3 + 2)(x^4 - 1) \mod 5 \end{aligned}$$

For each solution, we can consider the reduction modulo 5. Now by Fermat's Little Theorem, since 5 is prime, the solutions to $x^4 - 1 \mod 5$ are $1, 2, 3, 4 \mod 5$.

And if $x \equiv 0 \mod 5, (x^3 + 2)(x^4 - 1) = -2 \equiv 3 \mod 5$. So $x = 0$ is not a solution. So the only positive integers not solutions are a multiple of 5.

There are $1 + \left\lfloor \frac{2019}{5} \right\rfloor = 1 + 403 = 404$ such integers from 0 to 2019.

6. The product of all the solutions to the equation $x^3 - 10x^2 - x + 6 = 0$ is

   A. -6 B. -3 C. 1 D. 3 E. 6

Solution: A. -6

Let x_1, x_2, x_3 be the solutions. Then $x^3 - 10x^2 - x + 6 = (x - x_1)(x - x_2)(x - x_3)$. Note that the constant term in this product is $(-x_1)(-x_2)(-x_3) = -x_1x_2x_3 = 6$. So $x_1x_2x_3 = -6$.

7. Simplify $\tan(\arcsin(x))$.

   A. $\frac{x}{\sqrt{1-x^2}}$ B. $\frac{\sqrt{1-x^2}}{x}$ C. $\frac{1}{x}$ D. $\frac{1}{\sqrt{1-x^2}}$ E. $\sqrt{1-x^2}$

Solution: A. $\frac{x}{\sqrt{1-x^2}}$

Let $\arcsin(x) = \theta$, so that $\sin(\theta) = \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}}$. Solving for the adjacent side, we have adjacent= $\sqrt{\text{hypotenuse}^2 - \text{opposite}^2} = \sqrt{1-x^2}$.

So

$$\tan(\arcsin(x)) = \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}.$$

8. Perform the indicated operation. Simplify completely. Note k represents a positive integer.

$$\frac{x^{2k} - 16}{4x^{k+5} + 16x^5} \div \frac{x^{k+3} - 4x^3}{12x^{k+8}}$$

   A. $3k$ B. $\frac{1}{3}x^{k+15}$ C. $3x^k$ D. $\frac{3x^k(x^k+4)}{x^k+16}$ E. $\frac{1}{3x^k}$

Solution: C. $3x^k$

We have

$$\frac{x^{2k} - 16}{4x^{k+5} + 16x^5} \div \frac{x^{k+3} - 4x^3}{12x^{k+8}} = \frac{x^{2k} - 16}{4x^{k+5} + 16x^5} \times \frac{12x^{k+8}}{x^{k+3} - 4x^3}$$

Factoring we have

$$\frac{(x^k+4)(x^k-4)12x^kx^3x^5}{4x^5(x^k+4)x^3(x^k-4)}$$

Simplifying we have $3x^k$

9. You roll a six-sided die 3 times. How many ways can you roll so there is at least one 5?

   A. 31 B. 41 C. 81 D. 91 E. 125

Solution: D. 91

We know there are a possible 6^3 ways to roll the dice. Consider not rolling any 5's. We can do this is 5^3 ways. So the total number of ways to roll at least one 5 is $6^3 - 5^3 = 216 - 125 = 91$.

10. Let x, y, and z be relatively prime integers. Find $\gcd(x^{12}, y^{20}, z^{16}) \cdot \text{lcm}(x^{12}, y^{20}, z^{16})$.
A. $(xyz)^{\gcd(12,20,16)}$ B. $(xyz)^{\text{lcm}(12,20,16)}$ C. $(xyz)^{\text{lcm}(12,20,16)\gcd(12,20,16)}$
D. $x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}$ E. $x^{3\text{lcm}(12,20,16)}y^{5\text{lcm}(12,20,16)}z^{4\text{lcm}(12,20,16)}$

Solution: D. $x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}$

Since x, y, and z are relatively prime, $\gcd(x^{12}, y^{20}, z^{16}) = 1$ and

$$\text{lcm}(x^{12}, y^{20}, z^{16}) = x^{12}y^{20}z^{16}.$$

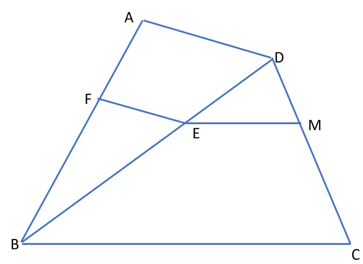
So

$$\gcd(x^{12}, y^{20}, z^{16}) \cdot \text{lcm}(x^{12}, y^{20}, z^{16}) = x^{12}y^{20}z^{16}.$$

Note $\gcd(12, 20, 16) = 4$, so we can rewrite this as

$$x^{12}y^{20}z^{16} = x^{3(4)}y^{5(4)}z^{4(4)} = x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}.$$

11. ⋆ ABCD is a quadrilateral. E is a point on the diagonal BD, EF is parallel to AD, and EM is parallel to BC. Assume that $\frac{FB}{AF} = \frac{3}{2}$, calculate $\frac{DM}{DC}$.



A. $\frac{2}{3}$ B. $\frac{2}{5}$ C. $\frac{3}{5}$ D. $\frac{3}{8}$ E. None of the above is correct

Solution: B. $\frac{2}{5}$

Since EF and AD are parallel, $\triangle BFE$ is similar to $\triangle BAD$. Thus $\frac{BE}{ED} = \frac{3}{2}$, or $\frac{ED}{BE} = \frac{2}{3}$.

And since EM is parallel to BC, $\triangle DEM$ is similar to $\triangle DBC$. So $\frac{ED}{BE} = \frac{DM}{MC} = \frac{2}{3}$.

Thus $\frac{DM}{DC} = \frac{2}{5}$.

12. How many three digit even numbers, have an even number of even digits?
A. 200 B. 225 C. 250 D. 400 E. 450

Solution: B. 225

Since we need an even number of even digits there are two cases to consider

Case 1: The hundreds digit and the ones digit are even. The hundreds digit can be 2,4,6,8, the ones digit can be 0,2,4,6,8, and the tens digit must be 1,3,5,7,9. So in total there are $4 \times 5 \times 5 = 100$ possible numbers.

Case 2: The tens digit and the ones digit are even. The tens digit can be 0, 2,4,6,8, the ones digit can be 0,2,4,6,8, and the hundreds digit must be 1,3,5,7,9. So in total there are $5 \times 5 \times 5 = 125$ possible numbers.

So in total there are 225.

13. How many different ways can the letters of $PROBABILITY$ be arranged if P and Y cannot be adjacent?

A. $9(10!)$ B. $10!$ C. $\frac{10(10!)}{4}$ D. $\frac{11(10!)}{4}$ E. $\frac{9(10!)}{4}$

Solution: E. $\frac{9(10!)}{4}$

The number of distinct ordered arrangements of all of the letters in $PROBABILITY$ is $\frac{11!}{2!2!}$, since there are 11 letters in total and two of each of the letters B and I.

Now attaching the P and the Y, in either order, the number of distinct ordered arrangements of all of the letters with P adjacent to Y is $2 \cdot \frac{10!}{2!2!} = \frac{10!}{2!}$.

Thus, the number of different ways to arrange the letters with P not adjacent to Y is the total number of ways to arrange the letters minus the number of ways in which P and Y are adjacent, which is $\frac{11!}{2!2!} - \frac{10!}{2!} = \frac{11! - 2 \cdot 10!}{2!2!} = \frac{10!(11-2)}{2!2!} = \frac{9(10!)}{4}$.

14. $\star$ The number $11^{10} - 1$ is divisible by which of the following numbers?

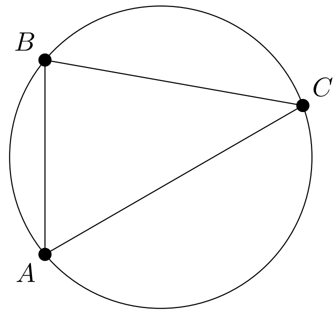
A. 11 B. 100 C. 121 D. 110 E. Both A. and B.

Solution: B. 100
From elementary algebra,

$$11^{10} - 1 = (11-1)(11^9 + 11^8 + 11^7 + 11^6 + 11^5 + 11^4 + 11^3 + 11^2 + 11 + 1).$$

The second factor on the right hand side is divisible by 10 because it is equal to a sum of 10 terms each of which ends with 1. Thus, $11^{10} - 1$ is equal to the product of 10 and a number divisible by 10, and consequently the difference $11^{10} - 1$ is divisible by 100.

15. Two ships located at Point A must travel to Point C. Ship 1 will travel straight to Point C at an average rate of 16 mph. Ship 2 will travel directly to Point B, a distance of 5 miles, and then directly to Point C. Assume $\angle BAC = 60°, \cos(\angle ABC) = \frac{1}{7}$, and the diameter of the triangle's circumcircle equals $\frac{14}{\sqrt{3}}$. If the ships leave Point A at the same time, what speed must Ship 2 average so that it arrives at Point C at the same time as Ship A?



A. 21 mph B. 22 mph C. 23 mph D. 24 mph E. None of these

Solution: D. 24 mph

Using the Law of Sines, $\overline{BC} = 7$ and the Law of Cosines, $\overline{AC} = 8$.

For Ship 1: $d = rt$, so $8 = 16t$. So it travels to Point C in $\frac{1}{2}$ hr

For Ship 2: It must travel: $5 + \overline{BC} = 5 + 7 = 12$ miles. So $d = rt$ and $12 = r\frac{1}{2}$. They must travel at an average speed of 24 mph.

16. ABC is a right triangle with $\angle C = 90^o$. Which of the following is NOT necessarily true?

A. $\cos A = \cos B$ B. $\sin(C - B) = \cos B$ C. $\sin(A + B) = 1$ D. $\cos(A + C) < 0$ E. $\cos(A + C) \cdot \cos(A - B) < 0$

Solution: A. $\cos A = \cos B$
From the given information we know $A + B = 90^o = C$. Hence, $A = C - B$.

We also have $\sin A = \sin(90^o - B) = \cos B$. So $\sin(C - B) = \cos B$. Thus, answer B is correct.

Since $A + B = 90^o$, $\sin(A + B) = 1$, so answer C is correct.

Additionally, $90^o < A + C < 180^o$, so $\cos(A + C) < 0$, showing that answer D is correct.

Finally, $-90^o < A - B < 90^o$, so $\cos(A - B) > 0$. Also, $90^o < A + C < 180^o$, so $\cos(A + C) < 0$. Therefore, $\cos(A + C) \cdot \cos(A - B) < 0$. Hence, answer E is correct.

Therefore, answer A is the only answer that is not necessarily correct.

17. The Russells have exactly four children, at least one of which is a girl. What is the probability that they have at least two boys? Assume that each child is equally likely to be a girl or a boy.

A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{4}$ E. $\frac{5}{6}$

Solution: C. $\frac{2}{3}$

Let X denote the number of girls among a family of four children. Then X has a Binomial$(4, 0.5)$ distribution. Thus, the probability of having no girls, $P(X = 0) = \frac{1}{16}$.

Similarly, $P(X = 1) = \frac{4}{16}$, $P(X = 2) = \frac{6}{16}$, $P(X = 3) = \frac{4}{16}$, and $P(X = 4) = \frac{1}{16}$.

Let A be the event of having at least one girl. Let B be the event of having at least two boys.

Then we are looking for the conditional probability of B given A, which is given by

$P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P((X=1) \cup (X=2))}{P(A)} = \dfrac{10/16}{15/16} = \dfrac{2}{3}$.

18. Suppose a math club has 30 members, in which there are 14 seniors and 16 juniors. In how many ways can a team of 10 of these students be chosen to compete at a math contest if there must be an equal number of seniors and juniors represented?

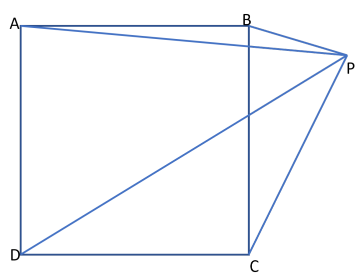
A. $\dfrac{14!16!}{9!11!}$ B. $\dfrac{14!16!}{5!9!11!}$ C. $\dfrac{14!16!}{5!5!9!11!}$ D. $\dfrac{14!16!}{10!9!11!}$ E. None of these

Solution: C. $\dfrac{14!16!}{5!5!9!11!}$

We wish to choose 5 out of 14 seniors to be on the team, where different orders of a choice of the same five people does not result in distinct teams. This gives a combination $\binom{14}{5}$. Similarly the number of ways to choose 5 juniors is $\binom{16}{5}$.

Thus, the number of possible teams is $\binom{14}{5} \cdot \binom{16}{5}$. This gives $\dfrac{14!}{5!9!} \cdot \dfrac{16!}{5!11!} = \dfrac{14!16!}{5!5!9!11!}$.

19. P is a point outside square ABCD, and closer to BC as indicated by the graph below. If $PA = \sqrt{15}$, $PB = \sqrt{2}$, and $PC = \sqrt{5}$, the length of PD is



A. $2\sqrt{5}$ B. $\sqrt{17}$ C. $3\sqrt{2}$ D. $2\sqrt{3}$ E. 0

Solution: C. $3\sqrt{2}$
Let E be a point on AD so that $AB \parallel EP$. Let EP intersect BC at point F. Let $\overline{AB} = a$, $\overline{AE} = b$, and $\overline{PF} = c$.

Since $\triangle AEP$ is a right triangle, we know that

$$(a+c)^2 + b^2 = 15. \tag{1}$$

With right triangle $\triangle BFP$ we have

$$c^2 + b^2 = 2. \tag{2}$$

With right triangle $\triangle CFP$ we have
$$(a-b)^2 + c^2 = 5. \tag{3}$$
With right triangle $\triangle PED$ we have
$$(\overline{PD})^2 = (a+c)^2 + (a-b)^2. \tag{4}$$

Adding equations (1) and (3) yields $(a+c)^2 + (a-b)^2 + b^2 + c^2 = 20$.
Applying equation (2), we have $(a+c)^2 + (a-b)^2 = 20 - 2 = 18$.
Therefore, $\overline{PD} = \sqrt{18} = 3\sqrt{2}$.

20. $\star$ Compute ln(2019) correct to the nearest integer.

A. 4 B. 6 C. 8 D. 10 E. 12

Solution: C. 8

$\ln(2019) \approx 7.61$, so the correct answer is 8. Note that $\ln(2) \approx 0.7$. Since $2048 = 2^{11}$, we have
$$\ln(2048) = \ln(2^{11}) = 11\ln(2) \approx 7.7.$$
Since 2019 is "close" to 2048, we expect ln(2019) to be "close" to 7.7, though slightly less than 7.7. Thus 8 is the best answer.

21. Five green balls and four red balls are in a bucket from which three balls are randomly selected without replacement. Find the probability that exactly two of the balls selected are red given that at least one red ball is selected.

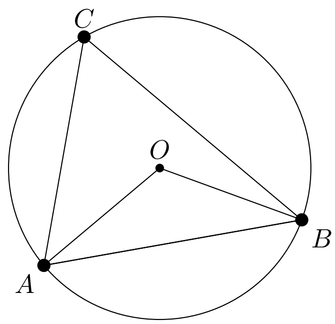
A. $\frac{15}{42}$ B. $\frac{15}{37}$ C. $\frac{37}{42}$ D. $\frac{37}{74}$ E. None of these

Solution: B. $\frac{15}{37}$
The probability that exactly two of the balls selected are red given that at least one red ball is selected is the probability that exactly 2 are red divided by the probability that at least one is red.

The probability that exactly 2 are red is $\frac{\binom{5}{1}\binom{4}{2}}{\binom{9}{3}} = \frac{30}{84}$, while the the probability that at least one is red is $\frac{\binom{9}{3} - \binom{5}{3}}{\binom{9}{3}} = \frac{84-10}{84} = \frac{74}{84}$.

Thus, the probability we are looking for is $\frac{\frac{30}{84}}{\frac{74}{84}} = \frac{30}{74} = \frac{15}{37}$.

22. An acute triangle ABC is inscribed inside a unit circle. If $\overline{AB} = \sqrt{3}$, $\angle ACB=$



A. 30^o B. 60^o C. 45^o D. 90^o E. 50^o

Solution: B. $60°$ Let D be the midpoint of $\overline{AB}$ and $\theta = \angle OAD$. Then $\overline{AD} = \frac{1}{2} \cdot \overline{AB} = \frac{\sqrt{3}}{2}$.

Now

$$\cos(\theta) = \frac{\overline{AD}}{\overline{AO}} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

So $\theta = 30°$.

Now $\triangle AOB$ is isosceles, so $\angle AOB = 120°$.

So by the inscribed angle theorem, $\angle ACB = 60°$.

23. For complex numbers z and w, if $\bar{z} - \bar{w} = 3$, find $z - w$.

A. -3 B. 3 C. $-3i$ D. $3i$ E. $3 + 3i$

Solution: B. 3

Since $\bar{z} - \bar{w}$ is a real number,

$$\overline{\bar{z} - \bar{w}} = 3.$$

So

$$z - w = \bar{\bar{z}} - \bar{\bar{w}} = \overline{\bar{z} - \bar{w}} = 3.$$

24. Consider three vectors $\vec{v} = \langle 1, -2, -3 \rangle, \vec{w} = \langle -1, -5, 3 \rangle$ and $\vec{z} = \langle 0, -4, -1 \rangle$. If the vector $\vec{p} = \langle 1, 0, 1 \rangle$, then express p in terms of $\vec{p} = a\vec{v} + b\vec{w} + c\vec{z}$ for some real numbers a, b, c.

A. $a = -3, b = -2, c = 4$ B. $a = 1, b = 2, c = -4$ C. $a = 4, b = 3, c = -4$ D. $a = 3, b = 2, c = -4$ E. None of these

Solution: D. $a = 3, b = 2, c = -4$ We want $\vec{p} = a\vec{v} + b\vec{w} + c\vec{z}$. That is,

$$\langle 1, 0, 1 \rangle = a\langle 1, -2, -3 \rangle + b\langle -1, -5, 3 \rangle + c\langle 0, -4, -1 \rangle$$

This gives the system:

$$\begin{aligned} a - b &= 1 \\ -2a - 5b - 4c &= 0 \\ -3a + 3b - c &= 1 \end{aligned}$$

From the first equation we see $a = b+1$. Plugging into the third equation we see that $-3b-3+3b-c = 1$. So $c = -4$. Plugging into the second equation we have $-2b - 2 - 5b - 4(-4) = 0$. So $-7b = -14$, $b = 2, a = 3$.

So $\vec{p} = 3\vec{v} + 2\vec{w} - 4\vec{z}$

25. Sally has the following money saving strategy.
On day #1 she will save \$1.
On day #2 she will save \$3 (given by \$1 + \$2).
On day #3 she will save \$6 (given by \$1 + \$2 + \$3).
Generally, on day n Sally will save $\$1 + \$2 + \cdots + \$n$.
How much money will Sally have saved in total after 20 days?

A. \$840 B. \$1190 C. \$1540 D. \$1890 E. \$2240

Solution: C. \$1540

Let T denote the total amount saved.

Then $T = \sum_{k=1}^{20}\sum_{j=1}^{k} j = \sum_{k=1}^{20} \frac{k(k+1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{20} k^2 + \sum_{k=1}^{20} k\right) = \frac{1}{2}\left(\frac{20\cdot 21\cdot 41}{6} + \frac{20\cdot 21}{2}\right) = 1540.$

26. Simplify

$$\frac{\sqrt[4]{x^3y}\sqrt{\frac{y}{x}}}{\sqrt[4]{\frac{y^3}{x}}}.$$

A. 1 B. $\sqrt{x}$ C. x D. $\sqrt{xy^3}$ E. $\sqrt{y^3}$

Solution: B. $\sqrt{x}$

We have

$$\frac{\sqrt[4]{x^3y}\sqrt{\frac{y}{x}}}{\sqrt[4]{\frac{y^3}{x}}} = \frac{x^{3/4}y^{1/4}\cdot y^{1/2}x^{-1/2}}{y^{3/4}x^{-1/4}}$$

$$= \frac{x^{1/4}y^{3/4}}{y^{3/4}x^{-1/4}} = x^{1/4}y^{3/4}y^{-3/4}x^{1/4} = x^{1/2} = \sqrt{x}$$

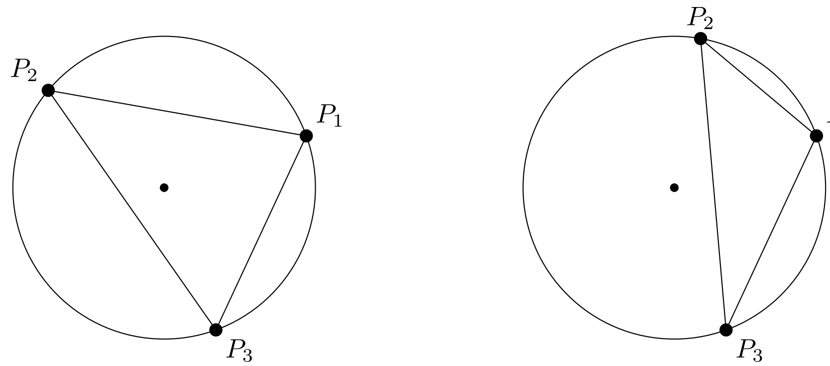
27. If $5\log_3 x = \log_3 y + 2$, express y as a function of x without the use of logarithms.

A. $y = \frac{1}{9}x^5$ B. $y = \frac{5}{9}x$ C. $y = \frac{1}{6}x^5$ D. $y = 5x - 9$ E. $y = \frac{1}{9}\sqrt[5]{x}$

Solution: A. $y = \frac{1}{9}x^5$

We see that $\log_3 x^5 = \log_3 y + 2$, so $\log_3 x^5 - \log_3 y = \log_3 \frac{x^5}{y} = 2$. Thus $\frac{x^5}{y} = 3^2$, and so $y = \frac{1}{9}x^5$.

28. ⋆ If three points are chosen randomly on a circle, what is the probability that the triangle formed by the points contains the circle's center?

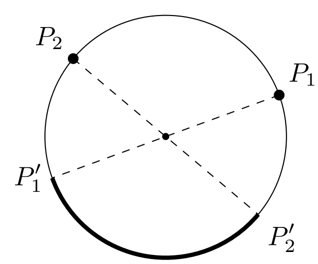


A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{4}$ E. $\frac{1}{4}$

Solution: E. $\frac{1}{4}$

First fix P_1 and P_2, and draw diameters from these points through the circle's center and meeting the circle again at points P_1' and P_2'.

Observe that when P_3 is chosen, the resulting triangle will contain the circle's center if and only if P_3 is contained in the arc between P_1' and P_2'. The probability that the triangle contains the circle's center is thus the length of this arc divided by the circle's total circumference.



Now, if we consider P_1 fixed and let P_2 vary, we observe that the arc from P_1' to P_2' can assume any length from 0 (if P_2 is placed very close to P_1) up to half the circle's circumference (if P_2 is placed opposite P_1). Since P_2 is chosen arbitrarily, the *average* length of the arc from P_1' to P_2' is one fourth the circle's circumference. Thus, the probability that the triangle contains the circle's center is

$$\frac{\text{length of arc } P_1' \text{ to } P_2'}{\text{total circumference}} = \frac{1}{4}.$$

29. What digit is in the ones place of the following quantity?

$$(123)^3 + (456)^3 + (789)^3$$

A. 0 B. 1 C. 2 D. 3 E. 4

Solution: C. 2

When 123 is cubed, the value in the ones-place is 7, because $3^3 = 27$. Likewise, when 456 is cubed the value in the ones-place is 6, because $6^3 = 216$. And again, since $9^3 = 729$, when 789 is cubed there is a 9 in the ones-place. When these three cubes are added, the value in the ones-place is determined by $7 + 6 + 9 = 22$. Thus, $(123)^3 + (456)^3 + (789)^3$ has a 2 in the ones-place.

30. The number 2^{29} is nine digits long, and all nine digits are distinct. Which of the digits from 0 to 9 is missing?

A. 6 B. 3 C. 0 D. 7 E. 4

Solution: E. 4

As a calculator will tell you,

$$2^{29} = 536870912,$$

so the missing digit is 4. You can find this by straightforward computation, which might be done more quickly by first observing that

$$2^{29} = 2^{10}2^{10}2^9 = (1024)(1024)(512).$$

31. Metro Department Store's monthly sales (in million dollars) during the past 3 months were as follows.

Month (x)	1	2	3
Monthly sales (y)	15	23	31

Estimate Metro's monthly sales after one year.

A. 86 million dollars B. 103 million dollars C. 15 million dollars D. 237 million dollars
E. None of these

Solution: B. 103 million dollars
Since the monthly sales increased by 8 million dollars per month entering both the second and third months, we can use a linear model with slope 8. Then when $x = 0$ we would have a y-intercept of $15 - 8 = 7$. Plugging $x = 12$ months into $y = 8x + 7$, we get $y = 8(12) + 7 = 103$ million dollars.

32. Simplify $\dfrac{f(x+h) - f(x)}{h}$ for $f(x) = \dfrac{1-x}{2+x}$.

A. $\dfrac{3}{(2+x+h)(2+x)}$ B. $\dfrac{-3}{(2+x+h)(2+x)}$ C. $\dfrac{-1}{(2+x+h)(2+x)}$ D. $\dfrac{1}{(2+x+h)(2+x)}$
E. None of these

Solution: B. $\dfrac{-3}{(2+x+h)(2+x)}$

We have $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(1-x-h)/(2+x+h) - (1-x)/(2+x)}{h}$
$= \dfrac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)}$

$$= \frac{2 - 2x - 2h + x - x^2 - xh - 2 - x - h + 2x + x^2 + xh}{h(2 + x + h)(2 + x)}$$
$$= \frac{-3h}{h(2 + x + h)(2 + x)}$$
$$= \frac{-3}{(2 + x + h)(2 + x)}.$$

33. Find the range of the function given by $f(x) = 6 \arctan(5x^3)$.

A. $(-\pi, \pi)$ B. $[-\pi, \pi]$ C. $(-3\pi, 3\pi)$ D. $(-6\pi, 6\pi)$ E. All real numbers

Solution: C. $(-3\pi, 3\pi)$
Note that the range of $y = 5x^3$ is all real numbers and $\frac{-\pi}{2} < \arctan x < \frac{\pi}{2}$, for all real x. Thus, the range of $y = \arctan(5x^3)$ is also $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Therefore the range of $f(x) = 6 \arctan(5x^3)$ is $6\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, or $(-3\pi, 3\pi)$.

34. Let x and y be natural numbers such that $x = a^m b^n$ and $y = a^r b^s$ where a, b, m, n, r, s are all natural numbers. Find $\gcd(x, y)$ if $\gcd(a, b) = 1$.

A. $a^{\min(m,r)} b^{\min(n,s)}$ B. $a^{\min(n,r)} b^{\min(m,s)}$ C. $a^{\max(m,n)} b^{\max(r,s)}$ D. $a^{\max(m,r)} b^{\max(n,s)}$
E. None of these

Solution: A. $a^{\min(m,r)} b^{\min(n,s)}$
Let G denote $\gcd(x, y)$. Since $\gcd(a, b) = 1$, G consists of as many powers of a and b that appear in both x and y. Thus, the highest power of a that can appear as a factor of G is $\min(m, r)$. Similarly, the highest power of b that can be a factor if G is $\min(n, s)$. Hence, $G = a^{\min(m,r)} b^{\min(n,s)}$.

35. Find the value of $\sin^2 47^o + \sin^2 43^o + (\tan^2 42^o)(\tan^2 48^o)$.

A. 0 B. 1 C. 2 D. 3 E. $\frac{1}{4}$

Solution: C. 2
Since $\sin(90^o - x) = \cos x$, we have $\sin 47^o = \cos 43^o$.
Since $\tan(90^o - x) = \cot x$, we have $\tan 42^o = \cot 48^o$.
Applying the above facts, we have $\sin^2 47^o + \sin^2 43^o + \tan^2 42^o \cdot \tan^2 48^o$
$= \cos^2 43^o + \sin^2 43^o + \cot^2 48^o \cdot \tan^2 48^o$
$= 1 + 1 = 2$.