2019 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 5, 11, 14 , 20, 28

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. A square has an area of 49 square inches. If the same amount is added to the length and removed from the width, the resulting rectangle has an area of 45 square inches. Find the dimensions of the rectangle in inches.

   A. 5×9 B. 4×9 C. 3×4 D. 5×10 E. 5×6

2. What is the smallest positive integer that is divisible by the first 12 composite numbers?

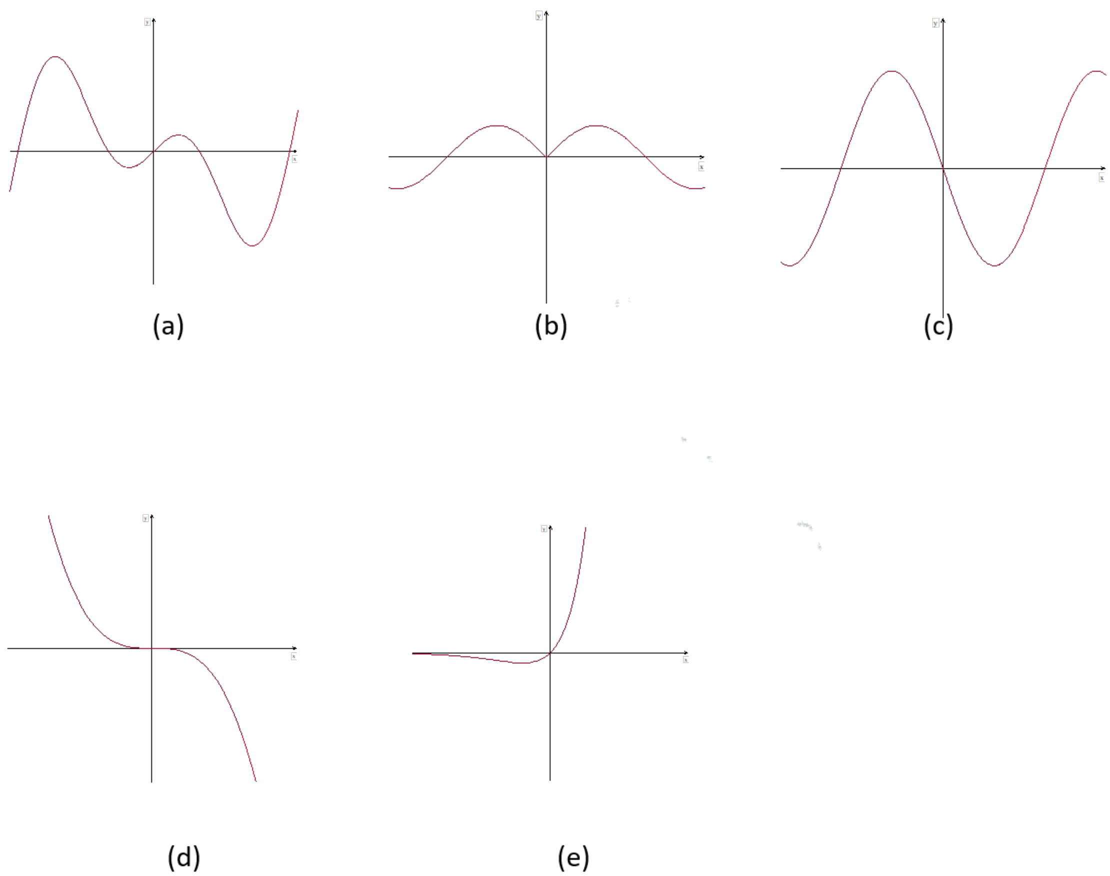
   A. 840 B. 10080 C. 2520 D. 5040 E. 27720

3. A boy who runs 8 miles per hour is 3/8 of the way through a railroad tunnel when he hears a train whistle behind him. If he runs back, he will leave the tunnel precisely the moment the train enters it. If instead he keeps on running to the far end of the tunnel, the train will reach him just as he leaves the tunnel.

   What is the speed of the train in miles per hour?

   A. 24 B. 28 C. 32 D. 36 E. 40

4. Assume f(x) is an odd function. Which of the following is the graph of $y = -xf(x)$?



5. ⋆ How many nonnegative integers less than 2019 are not solutions to $x^7 + 2x^4 - x^3 + 3 \equiv 0 \mod 5$?

   A. 402 B. 403 C. 404 D. 405 E. None of These

6. The product of all the solutions to the equation $x^3 - 10x^2 - x + 6 = 0$ is

   A. -6 B. -3 C. 1 D. 3 E. 6

7. Simplify $\tan(\arcsin(x))$.

   A. $\dfrac{x}{\sqrt{1-x^2}}$ B. $\dfrac{\sqrt{1-x^2}}{x}$ C. $\dfrac{1}{x}$ D. $\dfrac{1}{\sqrt{1-x^2}}$ E. $\sqrt{1-x^2}$

8. Perform the indicated operation. Simplify completely. Note k represents a positive integer.

$$\frac{x^{2k}-16}{4x^{k+5}+16x^5} \div \frac{x^{k+3}-4x^3}{12x^{k+8}}$$

   A. $3k$ B. $\dfrac{1}{3}x^{k+15}$ C. $3x^k$ D. $\dfrac{3x^k(x^k+4)}{x^k+16}$ E. $\dfrac{1}{3x^k}$

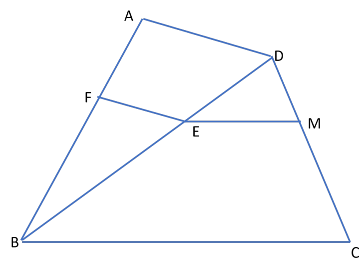
9. You roll a six-sided die 3 times. How many ways can you roll so there is at least one 5?

   A. 31 B. 41 C. 81 D. 91 E. 125

10. Let x, y, and z be relatively prime integers. Find $\gcd(x^{12}, y^{20}, z^{16}) \cdot \text{lcm}(x^{12}, y^{20}, z^{16})$.

   A. $(xyz)^{\gcd(12,20,16)}$ B. $(xyz)^{\text{lcm}(12,20,16)}$ C. $(xyz)^{\text{lcm}(12,20,16)\gcd(12,20,16)}$
   D. $x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}$ E. $x^{3\text{lcm}(12,20,16)}y^{5\text{lcm}(12,20,16)}z^{4\text{lcm}(12,20,16)}$

11. ⋆ ABCD is a quadrilateral. E is a point on the diagonal BD, EF is parallel to AD, and EM is parallel to BC. Assume that $\dfrac{FB}{AF} = \dfrac{3}{2}$, calculate $\dfrac{DM}{DC}$.



   A. $\dfrac{2}{3}$ B. $\dfrac{2}{5}$ C. $\dfrac{3}{5}$ D. $\dfrac{3}{8}$ E. None of the above is correct

12. How many three digit even numbers, have an even number of even digits?

   A. 200 B. 225 C. 250 D. 400 E. 450

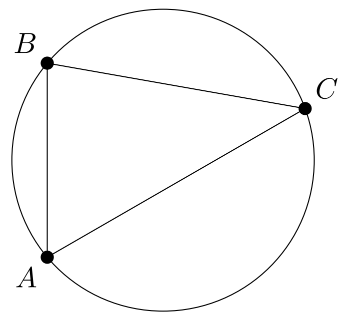
13. How many different ways can the letters of $PROBABILITY$ be arranged if P and Y cannot be adjacent?

   A. 9(10!) B. 10! C. $\dfrac{10(10!)}{4}$ D. $\dfrac{11(10!)}{4}$ E. $\dfrac{9(10!)}{4}$

14. ⋆ The number $11^{10} - 1$ is divisible by which of the following numbers?

   A. 11 B. 100 C. 121 D. 110 E. Both A. and B.

15. Two ships located at Point A must travel to Point C. Ship 1 will travel straight to Point C at an average rate of 16 mph. Ship 2 will travel directly to Point B, a distance of 5 miles, and then directly to Point C. Assume $\angle BAC = 60^\circ, \cos(\angle ABC) = \frac{1}{7}$, and the diameter of the triangle's circumcircle equals $\frac{14}{\sqrt{3}}$. If the ships leave Point A at the same time, what speed must Ship 2 average so that it arrives at Point C at the same time as Ship A?



A. 21 mph B. 22 mph C. 23 mph D. 24 mph E. None of these

16. ABC is a right triangle with $\angle C = 90^o$. Which of the following is NOT necessarily true?

A. $\cos A = \cos B$ B. $\sin(C - B) = \cos B$ C. $\sin(A + B) = 1$ D. $\cos(A + C) < 0$
E. $\cos(A + C) \cdot \cos(A - B) < 0$

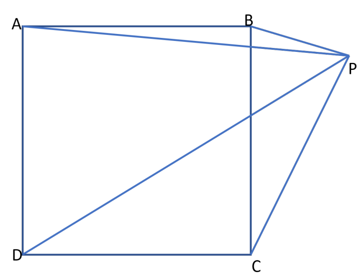
17. The Russells have exactly four children, at least one of which is a girl. What is the probability that they have at least two boys? Assume that each child is equally likely to be a girl or a boy.

A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{4}$ E. $\frac{5}{6}$

18. Suppose a math club has 30 members, in which there are 14 seniors and 16 juniors. In how many ways can a team of 10 of these students be chosen to compete at a math contest if there must be an equal number of seniors and juniors represented?

A. $\frac{14!16!}{9!11!}$ B. $\frac{14!16!}{5!9!11!}$ C. $\frac{14!16!}{5!5!9!11!}$ D. $\frac{14!16!}{10!9!11!}$ E. None of these

19. P is a point outside square ABCD, and closer to BC as indicated by the graph below. If $PA = \sqrt{15}$, $PB = \sqrt{2}$, and $PC = \sqrt{5}$, the length of PD is



A. $2\sqrt{5}$ B. $\sqrt{17}$ C. $3\sqrt{2}$ D. $2\sqrt{3}$ E. 0

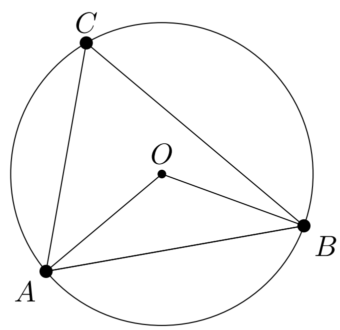
20. $\star$ Compute $\ln(2019)$ correct to the nearest integer.

    A. 4 B. 6 C. 8 D. 10 E. 12

21. Five green balls and four red balls are in a bucket from which three balls are randomly selected without replacement. Find the probability that exactly two of the balls selected are red given that at least one red ball is selected.

    A. $\frac{15}{42}$ B. $\frac{15}{37}$ C. $\frac{37}{42}$ D. $\frac{37}{74}$ E. None of these

22. An acute triangle ABC is inscribed inside a unit circle. If $\overline{AB} = \sqrt{3}$, $\angle ACB$=



    A. 30^o B. 60^o C. 45^o D. 90^o E. 50^o

23. For complex numbers z and w, if $\bar{z} - \bar{w} = 3$, find $z - w$.

    A. -3 B. 3 C. $-3i$ D. $3i$ E. $3 + 3i$

24. Consider three vectors $\vec{v} = \langle 1, -2, -3 \rangle, \vec{w} = \langle -1, -5, 3 \rangle$ and $\vec{z} = \langle 0, -4, -1 \rangle$. If the vector $\vec{p} = \langle 1, 0, 1 \rangle$, then express p in terms of $\vec{p} = a\vec{v} + b\vec{w} + c\vec{z}$ for some real numbers a, b, c.

    A. $a = -3, b = -2, c = 4$ B. $a = 1, b = 2, c = -4$ C. $a = 4, b = 3, c = -4$ D. $a = 3, b = 2, c = -4$ E. None of these

25. Sally has the following money saving strategy.
    On day #1 she will save \$1.
    On day #2 she will save \$3 (given by \$1 + \$2).
    On day #3 she will save \$6 (given by \$1 + \$2 + \$3).
    Generally, on day n Sally will save $\$1 + \$2 + \cdots + \$n$.
    How much money will Sally have saved in total after 20 days?

    A. \$840 B. \$1190 C. \$1540 D. \$1890 E. \$2240

26. Simplify

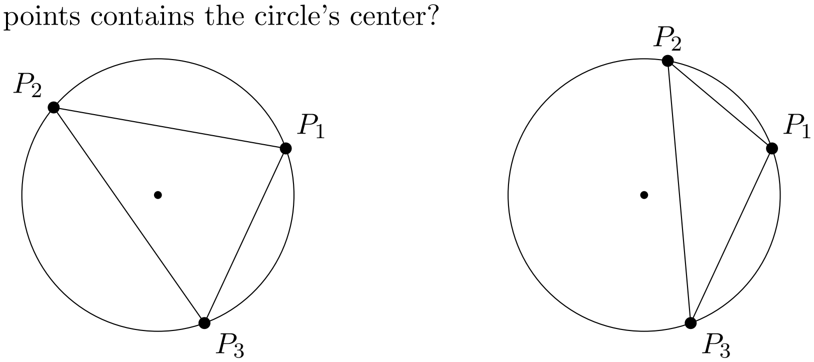
$$\frac{\sqrt[4]{x^3 y}\sqrt{\frac{y}{x}}}{\sqrt[4]{\frac{y^3}{x}}}.$$

    A. 1 B. $\sqrt{x}$ C. x D. $\sqrt{xy^3}$ E. $\sqrt{y^3}$

27. If $5\log_3 x = \log_3 y + 2$, express y as a function of x without the use of logarithms.

    A. $y = \frac{1}{9}x^5$ B. $y = \frac{5}{9}x$ C. $y = \frac{1}{6}x^5$ D. $y = 5x - 9$ E. $y = \frac{1}{9}\sqrt[5]{x}$

28. $\star$ If three points are chosen randomly on a circle, what is the probability that the triangle formed by the points contains the circle's center?



A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{4}$ E. $\frac{1}{4}$

29. What digit is in the ones place of the following quantity?

$$(123)^3 + (456)^3 + (789)^3$$

A. 0 B. 1 C. 2 D. 3 E. 4

30. The number 2^{29} is nine digits long, and all nine digits are distinct. Which of the digits from 0 to 9 is missing?

A. 6 B. 3 C. 0 D. 7 E. 4

31. Metro Department Store's monthly sales (in million dollars) during the past 3 months were as follows.

Month (x)	1	2	3
Monthly sales (y)	15	23	31

Estimate Metro's monthly sales after one year.

A. 86 million dollars B. 103 million dollars C. 15 million dollars D. 237 million dollars
E. None of these

32. Simplify $\dfrac{f(x+h)-f(x)}{h}$ for $f(x) = \dfrac{1-x}{2+x}$.

A. $\dfrac{3}{(2+x+h)(2+x)}$ B. $\dfrac{-3}{(2+x+h)(2+x)}$ C. $\dfrac{-1}{(2+x+h)(2+x)}$ D. $\dfrac{1}{(2+x+h)(2+x)}$
E. None of these

33. Find the range of the function given by $f(x) = 6\arctan(5x^3)$.

A. $(-\pi, \pi)$ B. $[-\pi, \pi]$ C. $(-3\pi, 3\pi)$ D. $(-6\pi, 6\pi)$ E. All real numbers

34. Let x and y be natural numbers such that $x = a^m b^n$ and $y = a^r b^s$ where a, b, m, n, r, s are all natural numbers. Find $\gcd(x, y)$ if $\gcd(a, b) = 1$.

A. $a^{\min(m,r)} b^{\min(n,s)}$ B. $a^{\min(n,r)} b^{\min(m,s)}$ C. $a^{\max(m,n)} b^{\max(r,s)}$ D. $a^{\max(m,r)} b^{\max(n,s)}$
E. None of these

35. Find the value of $\sin^2 47^o + \sin^2 43^o + (\tan^2 42^o)(\tan^2 48^o)$.

A. 0 B. 1 C. 2 D. 3 E. $\frac{1}{4}$