2018 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 9, 13, 17, 22, 34

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Solve the equation $x^2 - 1007x - 2018 = 0$.

   A. $-1007, -2018$ B. $-2, 1009$ C. $2, -1009$ D. $\dfrac{1007 \pm \sqrt{1005977}}{2}$ E. None of these

> **Solution: B** $-2, 1009$
> We have
> $$x^2 - 1007x - 2018 = (x - 1009)(x + 2) = 0.$$
> So $x = -2, 1009$

2. Consider a linear equation $y = -\sqrt{3}x + \dfrac{3}{4}$. If we increase x by 5 units, by how many units does the y-value change?

   A. $5\sqrt{3}$ B. $-5\sqrt{3}$ C. $-5\sqrt{3} + \dfrac{15}{4}$ D. $5\sqrt{3} + \dfrac{15}{4}$ E. $5\sqrt{3} - \dfrac{15}{4}$

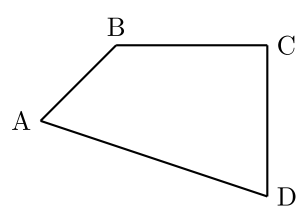
> **Solution: B.** $-5\sqrt{3}$
> For every one unit x increases by y decreases by $\sqrt{3}$. So if x is increased by 5 units, y will have changed by $-5\sqrt{3}$.

3. Among 2700 freshmen pursuing a business degree at a university, 1560 are enrolled in an economics course, 780 are enrolled in a mathematics course, and 340 are enrolled in both an economics and a mathematics course. What is the probability that a freshman selected at random from this group is enrolled in neither an economics course nor a mathematics course?

   A. $\dfrac{20}{27}$ B. $\dfrac{236}{270}$ C. $\dfrac{234}{270}$ D. $\dfrac{7}{27}$ E. None of these

> **Solution: D.** $\dfrac{7}{27}$
> We have $P(E \cup M) = P(E) + P(M) - P(E \cap M) = \dfrac{2000}{2700}$. Hence, $P[(E \cup M)^c] = 1 - \dfrac{2000}{2700} = \dfrac{700}{2700} = \dfrac{7}{27}$.

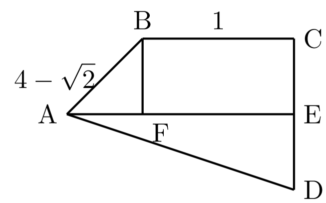
4. For a given quadrilateral ABCD, the length of $\overline{AB}$ is $4 - \sqrt{2}$, the length of $\overline{BC}$ is 1, the length of $\overline{CD}$ is 3. You are also given $\angle ABC = 135^o$ and $\angle BCD = 90^o$. Calculate the length of $\overline{AD}$.



   A. $2\sqrt{2}$ B. 3 C. $4\sqrt{2\sqrt{2}}$ D. $\dfrac{3}{2}$ E. $4\sqrt{2 - \sqrt{2}}$

Solution: E. $4\sqrt{2-\sqrt{2}}$

Let E be the point on $\overline{CD}$ satisfying $\overline{AE} \perp \overline{CD}$. Let F be the point on $\overline{AE}$ satisfying $\overline{AE} \perp \overline{BF}$. Then $\angle ABF = 45^o$, so $\overline{BF} = \overline{AF} = 4-\sqrt{2}\sin(45^o) = 2\sqrt{2}-1$. We have $\overline{AD}^2 = (1+\overline{AF})^2+\overline{ED}^2 = (1+\overline{AF})^2+(3-\overline{BF})^2 = (1+2\sqrt{2}-1)^2+(3-(2\sqrt{2}-1))^2 = 8+(4-2\sqrt{2})^2 = 8+16-16\sqrt{2}+8 = 16(2-\sqrt{2})$. Thus $\overline{AD} = 4\sqrt{2-\sqrt{2}}$.



5. If a and b are positive integers, then the number of pairs (a, b) that satisfy the equation $24a + b = ab$ is:

A. 2 B. 4 C. 6 D. 8 E. None of these

Solution: D. 8

When $a = 1$, we have $24 + b = b$, which has no solution.

Solving the equation for b, we have

$$b = \frac{24a}{a-1}.$$

When $a = 2, b = 48$ is one solution.

For positive integers $a > 2$, it is not possible for $a - 1$ to divide a. Since b is an integer, we must have that $a - 1$ divides 24. Thus $a - 1 = 1, 2, 3, 4, 6, 8, 12,$ or 24.

So

$$a = 2, 3, 4, 5, 7, 9, 13, 25$$

will each create a pair (a, b). So there are 8 pairs.

6. Simplify $\cos(2\arctan(x))$.

A. $\frac{x}{\sqrt{1+x^2}}$ B. $\frac{x}{\sqrt{1-x^2}}$ C. $\frac{1}{1-x^2}$ D. $\frac{1+x^2}{1-x^2}$ E. $\frac{1-x^2}{1+x^2}$

Solution: E. $\frac{1-x^2}{1+x^2}$

We have $\arctan(x) = \theta$, so that $\tan(\theta) = \frac{x}{1} = \frac{\text{opposite}}{\text{adjacent}}$. Solving for the hypothenuse, we have hypothenuse= $\sqrt{\text{adjacent}^2+\text{opposite}^2} = \sqrt{1+x^2}$.

So we have

$$\cos(2\arctan(x)) = \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 - \left(\frac{x}{\sqrt{1+x^2}}\right)^2$$

$$= \frac{1}{1+x^2} - \frac{x^2}{1+x^2} = \frac{1-x^2}{1+x^2}.$$

7. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$. Find the domain of $g \circ f$.
   A. $(0, 4)$ B. $[0, 4]$ C. $[4, \infty)$ D. $[-4, 4]$ E. $(-\infty, 4]$

Solution: B. $[0, 4]$
We have
$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2 - \sqrt{x}}.$$
For the domain we need, the domain of $f(x)$, which is $x \geq 0$. We then need $2 - \sqrt{x} \geq 0$ or $2 \geq \sqrt{x}$. So we have $4 \geq x$. Thus the domain of $g \circ f$ is $[0, 4]$.

8. Consider the equations $y = x^2 - 8x$ and $y = -\frac{1}{2}x^2 + mx - 3n$. If the equations have the same vertex, what are m and n?
   A. $m = 4, n = 8$ B. $m = -4, n = 8$ C. D. $m = 4, n = -8$ E. $m = -4, n = -8$ F. None of these

Solution: A. $m = 4, n = 8$

Given a quadratic $ax^2 + bx + c$, the vertex is at (h, k) where $h = \frac{-b}{2a}$ and $k = \frac{4ac - b^2}{4a}$.

The first equation tells us $h = \frac{8}{2} = 4$ and $k = \frac{4(1)(0) - (-8)^2}{4} = \frac{-64}{4} = -16$. So the vertex is at $(4, -16)$.

The second equation then tells us that
$$h = \frac{-m}{2(-1/2)} = m = 4$$
and
$$k = \frac{4(-1/2)(-3n) - m^2}{4(-1/2)} = \frac{6n - m^2}{-2} = \frac{6n - 4^2}{-2} = -16.$$
So $6n - 16 = 32, 6n = 48, n = 8$.

9. ($\star$) Find the last digit of the number 9^{9^9}.
   A. 0 B. 1 C. 3 D. 9 E. None of these

Solution: D. 9
Every even power of 9 can be represented in the form
$$9^{2n} = 81^n = \underbrace{81 \cdot 81 \cdot 81 \cdots 81}_{n \text{ times}}$$
and consequently its last digit is 1. Every odd power of 9 can be written in the form $9^{2n+1} = 9 \cdot 81^n$, and thus its last digit is 9. In particular, 9^{9^9} is an odd power of 9, and so its last digit is equal to 9.

10. For a given triangle ΔABC, $\angle ACB = 90^o$, $\angle ABC = 15^o$, and $\overline{BC} = 1$. Find the length of $\overline{AC}$.
A. $2+\sqrt{3}$ B. $2-\sqrt{3}$ C. $4-2\sqrt{3}$ D. $\sqrt{3}-\sqrt{2}$ E. $3-\sqrt{2}$

Solution: B. $2-\sqrt{3}$
Let $x = \overline{AC} = \tan 15^o$. Then $1+x^2 = 1+\tan^2 15^o = \dfrac{1}{\cos^2 15^o}$. We use the formula $\cos 2t = 2\cos^2 t - 1$ to see that $\cos^2 15^o = \dfrac{\cos 30^o + 1}{2} = \dfrac{\sqrt{3}/2+1}{2}$.
Therefore, $x^2 = \dfrac{2}{\sqrt{3}/2+1} - 1 = \dfrac{1-\sqrt{3}/2}{\sqrt{3}/2+1}$,
and $x = \sqrt{\dfrac{1-\sqrt{3}/2}{\sqrt{3}/2+1}} = \sqrt{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}} = \sqrt{\dfrac{(2-\sqrt{3})^2}{4-3}} = 2-\sqrt{3}$.

11. Define $S(n)$ as the sum of the digits of a positive integer n. If $S(n) = 2018$, what is the smallest possible value for $S(n+1)$?
A. 3 B. 4 C. 11 D. 12 E. 2019

Solution: A. 3
Note that there are multiple n that satisfy $S(n) = 2018$. Since adding one to n, to minimize $S(n+1)$ we would want an integer with the most possible 9's at the end due to carrying. We see that $2018 = 2 + 9 \cdot 224$, so we can let $n = 2\underbrace{9\cdots9}_{224\text{times}}$. Thus $n+1 = 3\underbrace{0\cdots0}_{224\text{ times}}$. Thus $S(n+1) = 3$.

12. A pharmacist wishes to select four brands of aspirin to sell in his store. He has seven major brands to choose from: A, B, C, D, E, F and G. If he selects four brands at random, what is the probability that he will select brand B and/or brand C?
A. $\frac{1}{7}$ B. $\frac{6}{7}$ C. $\frac{2}{7}$ D. $\frac{4}{7}$ E. $\frac{5}{7}$

Solution: B. $\frac{6}{7}$
First, note that the total number of possible selections is $C(7,4) = 35$. The number of ways to select brand B is the same as the number of ways to select brand C, which is $C(6,3) = 20$. The number of ways to select both brands B and C is $C(5,2) = 10$. Thus, probability that he will select brand B and/or brand C is $P(B \cup C) = P(B) + P(C) - P(B \cap C) = \dfrac{20}{35} + \dfrac{20}{35} - \dfrac{10}{35} = \dfrac{30}{35} = \dfrac{6}{7}$.

13. ($\star$) A teenage boy wrote his own age after his father's. From this new four digit number he subtracted the absolute value of the difference of their ages to get 4,289. Find the sum of their ages.
A. 48 B. 52 C. 56 D. 59 E. 64

Solution:
D. 59

Let b and f denote the boy's and the father's age, respectively. Then $100f + b - (f - b) = 4289$. That is $99f + 2b = 4289$. Equivalently, $99f = 4257 + 32 - 2b$, or $f - 43 = \frac{32-2b}{99}$. If $f \geq 44$ then $\frac{32-2b}{99} \geq 1$, and so $b < 0$, which is impossible. If $f \leq 42$, then $\frac{32-2b}{99} \leq -1$, which means $b > 65$, implying that the son is not a teenager. If $f = 43$, then $b = 16$ and the boy is indeed a teenager. Thus, the sum of their ages is $43 + 16 = 59$.

14. Consider the following:
    - On the 1^{st} day after Christmas, John gives his true love one present.
    - On the 2^{nd} day after Christmas, John gives his true love three additional presents $(1 + 2)$.
    - On the 3^{rd} day after Christmas, John gives his true love six additional presents $(1 + 2 + 3)$.
    - In general, on the k^{th} day after Christmas, John gives his true love $(1 + 2 + \cdots + k)$ additional presents.

    How many presents in total has John given to his true love after 30 days of giving?

    A. 3654 B. 4060 C. 4495 D. 4960 E. 5456

Solution: D. 4960
The total number of presents given

$$T = \sum_{k=1}^{30}\sum_{j=1}^{k} j = \sum_{k=1}^{30} \frac{k(k+1)}{2} = \frac{1}{2}\left(\sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k\right) = \frac{1}{2}\left(\frac{30 \cdot 31 \cdot 61}{6} + \frac{30 \cdot 31}{2}\right) = 4960.$$

15. Let $\vec{a}$ and $\vec{b}$ be two vectors. Suppose $|\vec{a}| = 3$ and $|\vec{b}| = 4$ and $|\vec{a} - \vec{b}| = \sqrt{13}$. Find the degree measurement of the angle between these two vectors. Assume the angle is between 0^o and 180^o.

    A. 135 B. 120 C. 30 D. 45 E. 60

Solution: E. 60
Answer: We have $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 3^2 - 2\vec{a} \cdot \vec{b} + 4^2 = 25 - 2\vec{a} \cdot \vec{b}$. As $|\vec{a} - \vec{b}|^2 = 13$, we get $\vec{a} \cdot \vec{b} = 6$. Now $cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{6}{12} = \frac{1}{2}$. Hence, $\theta = 60^o$.

16. How many real number solutions does the following equation have? $\sqrt{1 - \sqrt{1 + x}} = x$.

    A. 1 B. 2 C. 3 D. 4 E. None

Solution: A 1
Squaring both sides we have $1 - \sqrt{1 + x} = x^2$. Or $1 - x^2 = \sqrt{1 + x}$. Squaring both sides again we have

$$(1 - x^2)^2 = 1 - 2x^2 + x^4 = 1 + x.$$

So we want to solve

$$x^4 - 2x^2 - x = x(x^3 - 2x - 1) = 0.$$

We have $x(x^3-2x-1) = x(x^3-x-x-1) = x[x(x^2-1)-(x+1)] = x[x(x+1)(x-1)-(x+1)]$ $= x(x+1)[x(x-1)-1] = x(x+1)(x^2-x-1) = 0$.

Thus, the roots are $x = 0, -1, \dfrac{1 \pm \sqrt{5}}{2}$.

Substituting these back into the original equation $\sqrt{1-\sqrt{1+x}} = x$ we find that the only solution is $x = 0$. Therefore, there is exactly one solution.

17. ($\star$) A box contains red and blue marbles. The number of blue marbles is four times the number of red marbles. A marble will be randomly selected.

- If the selected marble is red, then it will be returned into the box along with 15 additional blue marbles.
- If the selected marble is blue, then it will be returned into the box; no additional marbles will be placed into the box.

A second marble will then be selected. Prior to selecting the first marble, the probability of the second selected marble being blue equals 81%. How many blue marbles were originally in the box?

A. 4 B. 16 C. 36 D. 56 E. 64

Solution: C. 36
Let x denote the number of red marbles originally in the box.
Then the original box contained x red marbles and $4x$ blue marbles, giving $5x$ marbles in total.
For $k = 1, 2$ let B_k denote the event that the kth selected marble was blue.
Let R_k denote the event that the kth selected marble was red.
Now $B_2 \equiv (R_1 \cap B_2) \cup (B_1 \cap B_2)$, with the two events in the union being mutually exclusive.
Thus $P(B_2) = P(R_1 \cap B_2) + P(B_1 \cap B_2)$
$= P(R_1)P(B_2|R_1) + P(B_1)P(B_2|B_1)$
$= \dfrac{1}{5} \cdot \dfrac{4x+15}{5x+15} + \left(\dfrac{4}{5}\right)^2$.
Setting the above equal to $\dfrac{81}{100}$ and solving for x yields $x = 9$. Therefore, the original box contained 36 blue marbles.

18. Determine the rational number represented by $2^{-3\log_2 5}$.

A. $\dfrac{1}{15}$ B. $\dfrac{1}{125}$ C. 0 D. -15 E. -125

Solution: B. $\dfrac{1}{125}$
Note that
$$2^{-3\log_2 5} = 2^{\log_2 5^{-3}} = 5^{-3} = \frac{1}{125}.$$

19. Let y and z be negative real numbers satisfying $\dfrac{1}{y} + \dfrac{1}{z} - \dfrac{1}{y-z} = 0$. Calculate $\dfrac{z}{y}$.

A. $\dfrac{1+\sqrt{5}}{2}$ B. $\dfrac{1-\sqrt{5}}{2}$ C. $\dfrac{-1+\sqrt{5}}{2}$ D. $\dfrac{-1-\sqrt{5}}{2}$ E. $\dfrac{\sqrt{5}}{2}$

Solution: C. $\dfrac{-1+\sqrt{5}}{2}$

Multiplying on both sides by y, we have

$$1+\frac{y}{z}-\frac{y}{y-z}=0.$$

We can rewrite this as

$$1+\frac{1}{\frac{z}{y}}-\frac{1}{1-\frac{z}{y}}=0.$$

Letting $x=\dfrac{z}{y}$, we have

$$1+\frac{1}{x}-\frac{1}{1-x}=0.$$

So

$$\frac{x+1}{x}=\frac{1}{1-x}\Rightarrow 1-x^2=x.$$

Solving $x^2+x-1=0$ we see $x=\dfrac{-1\pm\sqrt{5}}{2}$. Since y and z are negative, x must be positive, so $x=\dfrac{-1+\sqrt{5}}{2}$.

20. The number 2^{29} is nine-digits long, and all nine digits are distinct. Which of the digits from 0 to 9 is missing?

A. 3 B. 4 C. 6 D. 7 E. 9

Solution: B. 4

We have

$$2^{29}=2^{10}2^{10}2^{9}=(1024)(1024)(512)=536870912.$$

21. Find the 2018^{th} smallest x, with $x>1$ that satisfies the equation

$$\sin(\ln x)+2\cos(505\ln x)\sin(504\ln x)=0.$$

A. $e^{2018\pi}$ B. $e^{2\pi}$ C. 2018π D. e^{π} E. $e^{4\pi}$

Solution: B. $e^{2\pi}$

Let $y=\ln x$. Using trigonometric properties we have $2\cos(505y)\sin(504y)=\sin(505y+504y)-\sin(505y-504y)=\sin(1009y)-\sin(y)$. Thus we are solving

$$\sin(y)+2\cos(505y)\sin(504y)=\sin(y)+\sin(1009y)-\sin(y)=\sin(1009y)=0.$$

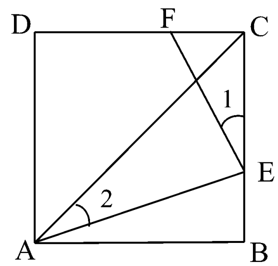
We know our solutions will be $y=\dfrac{n\pi}{1009}$ for $n>1$. That is

$$\ln x=\frac{n\pi}{1009}\Rightarrow x=e^{\frac{n\pi}{1009}}.$$

Since we want the 2018th positive solution, we have

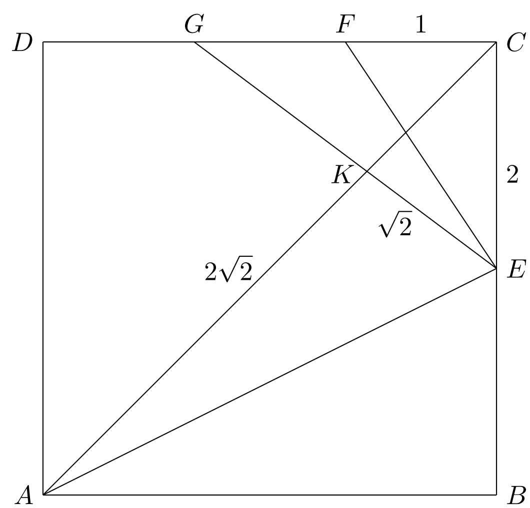
$$x=e^{\frac{2018\pi}{1009}}=e^{2\pi}.$$

22. (⋆) For a given square $ABCD$, F is a point on $\overline{CD}$ satisfying $\overline{CF} = \frac{1}{3}\overline{CD}$, and E is a point on $\overline{BC}$ satisfying $\overline{BE} = \frac{1}{3}\overline{BC}$. Which of the following is true?



A. $\angle CAE = \angle CEF$ B. $\angle CAE > \angle CEF$ C. $\angle CAE < \angle CEF$ D. $\angle CAE = \frac{1}{2}\angle CEF$
E. None of the above is correct.

Solution: **A.** $\angle CAE = \angle CEF$



Let $\overline{BC} = 3$, so that $\overline{CE} = 2$. Define G to be the point on $\overline{CD}$ satisfying $\overline{DG} = \frac{1}{3}\overline{CD}$. Define K to be the point of intersection between $\overline{GE}$ and $\overline{AC}$.

In $\triangle CKE$, $\angle CKE = 90^\circ$, so we have a 45-45-90 triangle. Thus the length of the legs are $\overline{CK} = \overline{KE} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

We see $\triangle ABC$ is a 45-45-90 triangle with legs of length 3. Thus $\overline{CA} = 3\sqrt{2}$. Therefore, $\overline{AK} = \overline{CA} - \overline{CK} = 2\sqrt{2}$.

Consider triangles $\triangle AKE$ and $\triangle ECF$. We see that $\frac{\overline{AK}}{\overline{KE}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$ and $\frac{\overline{EC}}{\overline{CF}} = \frac{2}{1} = 2$. And $\angle AKE = 90^\circ = \angle ECF$. Thus $\triangle AKE$ is similar to $\triangle ECF$. Therefore $\angle CEF = \angle KAE$. Since $\angle KAE = \angle CAE, \angle CAE = \angle CEF$.

23. A pair of fair dice is rolled. What is the probability the first die is a 3 given the sum is a 6?

A. $\frac{1}{12}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{5}$ E. $\frac{2}{5}$

Solution: D. $\frac{1}{5}$

Let A be the event the first die is a 3 and B the event the sum is 6. Now $P(B) = \frac{5}{36}$ since we can roll 1-5, 2-4, 3-3, 4-2, 5-1. And $P(A \cap B) = \frac{1}{36}$ since the only possible roll is 3-3. We want

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{5/36} = \frac{1}{5}$$

24. If the first day of March fell on a Tuesday one year, on what day of the week would August $15th$ fall on that same year?

A. Saturday B. Sunday C. Monday D. Tuesday E. Wednesday

Solution: C. Monday

One way to approach this is to determine how many days August $15th$falls after March $1st$. This difference is $30 + 30 + 31 + 30 + 31 + 15 = 167$. When 167 is divided by 7 the remainder is 6. So August $15th$ falls six days of the week after Tuesday, which is Monday.

25. Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} x & -1 \\ y & 1 \end{bmatrix}$. Find the values of real numbers x and y that satisfy $AB = BA$.

A. $x = \frac{5}{2}$, $y = \frac{-3}{2}$ B. $x = \frac{-3}{2}$, $y = \frac{-3}{2}$ C. $x = \frac{-3}{2}$, $y = \frac{5}{2}$ D. $x = \frac{5}{2}$, $y = \frac{3}{2}$ E. None of these

Solution: A. $x = \frac{5}{2}$ and $y = \frac{-3}{2}$

Multiplying, we find $AB = \begin{bmatrix} x + 2y & 1 \\ 3x + 4y & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} x - 3 & 2x - 4 \\ y + 3 & 2y + 4 \end{bmatrix}$. Thus, we must have $2x - 4 = 1$ and $2y + 4 = 1$, yielding $x = \frac{5}{2}$ and $y = \frac{-3}{2}$.

To check, we see $x + 2y = x - 3$ becomes $\frac{5}{2} + 2\left(\frac{-3}{2}\right) = -\frac{1}{2} = \frac{5}{2} - 3$. And $3x + 4y = y + 3$ becomes $3\left(\frac{5}{2}\right) + 4\left(\frac{-3}{2}\right) = \frac{3}{2} = \frac{-3}{2} + 3$.

26. Assume $\frac{1}{a} - |a| = 1$ for a real number a. Find out the value of $\frac{1}{a} + |a|$.

A. $\frac{\sqrt{5}}{2}$ B. $\frac{-\sqrt{5}}{2}$ C. $\sqrt{5}$ D. $-\sqrt{5}$ E. 0

Solution: C. $\sqrt{5}$

Let $a > 0$. Then $\frac{1}{a} - |a| = \frac{1}{a} - a = 1$. Squaring we have

$$\frac{1}{a^2} - 2 + a^2 = 1.$$

Adding 4 to both sides we have

$$\frac{1}{a^2} + 2 + a^2 = \left(\frac{1}{a} + a\right)^2 = 5.$$

Taking the square root we see $\frac{1}{a} + a = \sqrt{5}$.

Now suppose $a < 0$. Then $\frac{1}{a} - |a| = \frac{1}{a} + a = 1$. Squaring both sides we have

$$\frac{1}{a^2} + 2 + a^2 = 1.$$

Subtracting 4 from both sides we have

$$\frac{1}{a^2} - 2 + a^2 = \left(\frac{1}{a} - a\right)^2 = -3.$$

This is impossible. So our solution is $\sqrt{5}$.

27. A container is filled with 8 liters of a 20% salt solution. How many liters of pure water must be added to produce a 15% salt solution?

A. $\frac{8}{3}$ B. $\frac{3}{5}$ C. $\frac{3}{8}$ D. 2 E. $\frac{13}{5}$

Solution: A. $\frac{8}{3}$

Let x denote the number of liters of water added to achieve a 15% salt solution. The amount of salt in the original solution is $0.2(8)$. After x liters of pure water are added, the amount of salt in the solution is $0.15(8+x)$. Thus, we have $0.2(8) = 0.15(8+x)$, giving $1.6 = 1.2 + 0.15x$. So $0.15x = 0.4$ and $x = \frac{40}{15} = \frac{8}{3}$.

28. The polynomial $p(n) = n^2 + n + 41$ has a curious property: for small integer values of n, $p(n)$ is prime. For example, $p(1) = 43$, which is prime, $p(2) = 47$, which is also prime, and $p(3) = 53$, which is once again prime. Find the smallest positive integer n such that $p(n)$ is *not* prime.

A. 37 B. 38 C. 39 D. 40 E. 41

Solution:

D. 40 The first "obvious" value of n for which $p(n)$ is not prime is $n = 41$, because

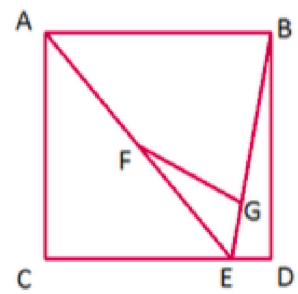
$$p(41) = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43.$$

That's not the answer, however, because

$$p(40) = 40^2 + 40 + 41 = 40(40 + 1) + 41 = 41^2.$$

As for why $p(n)$ is prime for every $n \le 39$, well, that's tricky to prove, and beyond tedious to check.

29. Assume the side length of square ABCD is 5. E is an arbitrary point on CD. F is the mid point of AE. G is a point on BE satisfying $BG = 3EG$. What is the area of triangle EFG?



A. 5/4 B. 5/6 C. 25/16 D. 3/2

Solution: C. 25/16

Area (EFG) = Area (EFB)/4 = Area (EAB) /2 /4 =$(5^2/2)/2/4$=25/16. Therefore (c) is correct.

30. Write $\dfrac{5-2i}{-3+i}$ in the form $a + bi$.

A. $\dfrac{-13}{10} + \dfrac{1}{10}i$ B. $\dfrac{-13}{8} - \dfrac{1}{8}i$ C. $\dfrac{-17}{8} + \dfrac{1}{8}i$ D. $\dfrac{-17}{10} + \dfrac{-1}{10}i$ E. $\dfrac{-17}{10} + \dfrac{1}{10}i$

Solution: E. $\dfrac{-17}{10} + \dfrac{1}{10}i$

We have $\dfrac{5-2i}{-3+i} = \dfrac{(5-2i)}{(-3+i)}\dfrac{(-3-i)}{(-3-i)} = \dfrac{-15-5i+6i-2}{9+1} = \dfrac{-17+i}{10}$.

31. Find the value of $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$.

A. -1 B. $2\sqrt[3]{2}$ C. $\sqrt[3]{2} - \sqrt[3]{2}$ D. 1 E. None of these

Solution: D. 1

Let $x = \sqrt{5} - 2$. Then note that $\sqrt{5} + 2 = x + 4$ and $x(x + 4) = (\sqrt{5} - 2)(\sqrt{5} + 2) = 1$. Now let $y = \sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$ and re-write this expression as $y + \sqrt[3]{x} = \sqrt[3]{x+4}$. Cube each side to obtain,

$$y^3 + 3y\sqrt[3]{x^2} + 3y^2\sqrt[3]{x} + x = x + 4,$$

which gives

$$y^3 - 4 = -3y\sqrt[3]{x}(y + \sqrt[3]{x}) = -3y\sqrt[3]{x}\sqrt[3]{x+4} = -3y\sqrt[3]{x(x+4)} = -3y.$$

Hence we find $y^3 + 3y - 4 = 0$, for which $y = 1$ is the only option that satisfies this equation.

32. A right-truncated prime number is a prime number that yields another prime number when you delete it's ones digit. For example 431 is a right-truncated prime because 43 is prime. How many two digit right-truncated primes are there?

A. 8 B. 9 C. 10 D. 12 E. 13

Solution: B. 9

We only need to check for primes in the double digits that begin with $2, 3, 5, 7$ since those are the single digit prime numbers. So we have $23, 29, 31, 37, 53, 59, 71, 73, 79$ as the right-truncated primes.

33. Our codebreakers have recently uncovered a top secret organization of spies who collaborate on nefarious plots. Their activities are still mostly hidden from us, but we have managed to uncover the following facts:

- Any two distinct plots involve exactly one spy in common.
- Every spy is involved in exactly two plots.
- There are exactly four plots.

Exactly how many spies are there?

A. 5 B. 6 C. 8 D. 12 E. Infinitely many

Solution: B. 6

There are exactly six spies!

We know there are exactly four plots, and any two plots have exactly one spy in common. As there are six possible pairs of plots, there must be (at least) six spies. If there were a seventh spy, there would be a plot with two common spies, which contradicts our facts.

34. ($\star$) How many integers between 50 and 500 are divisible by 11?

A. 40 B. 41 C. 39 D. 42 E. 43

Solution: B. 41

Let n denote the number of integers between 50 and 500 that are divisible by 11. From checking the divisibility of integers near 50 and 500, we see that the first such number is 55 and the nth (last) such number is 495. Thus, 495 can be formed by starting with 55 and adding 11 a total of $n-1$ times. Thus, we have $55 + (n-1)11 = 495$, which gives $55 + 11n - 11 = 495$. Thus, $11n = 495 - 44 = 451$. So $n = \dfrac{451}{11} = 41$.

Alternatively, the number of positive integers less than 500 that are divisible by 11 is $\lfloor 500/11 \rfloor = 45$. We subtract from this the number of integers less than 50 which are divisible by 11. These are 11, 22, 33, 44. Thus, there are 45-4=41 such integers.

35. Let x, y, and z be relatively prime integers. Which of the following expressions is equal to $\text{lcm}(x^{12}, y^{20}, z^{16})$?
   A. $(xyz)^{\gcd(12,20,16)}$ B. $(xyz)^{\text{lcm}(12,20,16)}$ C. $(xyz)^{\text{lcm}(12,20,16)\gcd(12,20,16)}$
   D. $x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}$ E. $x^{3\text{lcm}(12,20,16)}y^{5\text{lcm}(12,20,16)}z^{4\text{lcm}(12,20,16)}$

Solution: D. $x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}$

Since x, y, and z are relatively prime,

$$\text{lcm}(x^{12}, y^{20}, z^{16}) = x^{12}y^{20}z^{16}.$$

Note $\gcd(12, 20, 16) = 4$, so we can rewrite this as

$$x^{12}y^{20}z^{16} = x^{3(4)}y^{5(4)}z^{4(4)} = x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}.$$