2018 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 9, 13, 17, 22, 34

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Solve the equation $x^2 - 1007x - 2018 = 0$.

   A. $-1007, -2018$ B. $-2, 1009$ C. $2, -1009$ D. $\dfrac{1007 \pm \sqrt{1005977}}{2}$ E. None of these

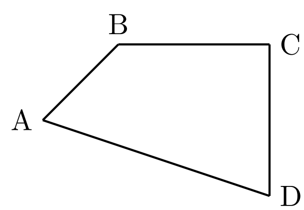
2. Consider a linear equation $y = -\sqrt{3}x + \dfrac{3}{4}$. If we increase x by 5 units, by how many units does the y-value change?

   A. $5\sqrt{3}$ B. $-5\sqrt{3}$ C. $-5\sqrt{3} + \dfrac{15}{4}$ D. $5\sqrt{3} + \dfrac{15}{4}$ E. $5\sqrt{3} - \dfrac{15}{4}$

3. Among 2700 freshmen pursuing a business degree at a university, 1560 are enrolled in an economics course, 780 are enrolled in a mathematics course, and 340 are enrolled in both an economics and a mathematics course. What is the probability that a freshman selected at random from this group is enrolled in neither an economics course nor a mathematics course?

   A. $\dfrac{20}{27}$ B. $\dfrac{236}{270}$ C. $\dfrac{234}{270}$ D. $\dfrac{7}{27}$ E. None of these

4. For a given quadrilateral ABCD, the length of $\overline{AB}$ is $4 - \sqrt{2}$, the length of $\overline{BC}$ is 1, the length of $\overline{CD}$ is 3. You are also given $\angle ABC = 135^o$ and $\angle BCD = 90^o$. Calculate the length of $\overline{AD}$.



   A. $2\sqrt{2}$ B. 3 C. $4\sqrt{2\sqrt{2}}$ D. $\dfrac{3}{2}$ E. $4\sqrt{2 - \sqrt{2}}$

5. If a and b are positive integers, then the number of pairs (a, b) that satisfy the equation $24a + b = ab$ is:

   A. 2 B. 4 C. 6 D. 8 E. None of these

6. Simplify $\cos(2\arctan(x))$.

   A. $\dfrac{x}{\sqrt{1+x^2}}$ B. $\dfrac{x}{\sqrt{1-x^2}}$ C. $\dfrac{1}{1-x^2}$ D. $\dfrac{1+x^2}{1-x^2}$ E. $\dfrac{1-x^2}{1+x^2}$

7. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$. Find the domain of $g \circ f$.

   A. $(0, 4)$ B. $[0, 4]$ C. $[4, \infty)$ D. $[-4, 4]$ E. $(-\infty, 4]$

8. Consider the equations $y = x^2 - 8x$ and $y = -\dfrac{1}{2}x^2 + mx - 3n$. If the equations have the same vertex, what are m and n?

   A. $m = 4, n = 8$ B. $m = -4, n = 8$ C. D. $m = 4, n = -8$ E. $m = -4, n = -8$
   F. None of these

9. ($\star$) Find the last digit of the number 9^{9^9}.

   A. 0 B. 1 C. 3 D. 9 E. None of these

10. For a given triangle ΔABC, $\angle ACB = 90^o$, $\angle ABC = 15^o$, and $\overline{BC} = 1$. Find the length of $\overline{AC}$.

    A. $2+\sqrt{3}$ B. $2-\sqrt{3}$ C. $4-2\sqrt{3}$ D. $\sqrt{3}-\sqrt{2}$ E. $3-\sqrt{2}$

11. Define $S(n)$ as the sum of the digits of a positive integer n. If $S(n) = 2018$, what is the smallest possible value for $S(n+1)$?

    A. 3 B. 4 C. 11 D. 12 E. 2019

12. A pharmacist wishes to select four brands of aspirin to sell in his store. He has seven major brands to choose from: A, B, C, D, E, F and G. If he selects four brands at random, what is the probability that he will select brand B and/or brand C?

    A. $\frac{1}{7}$ B. $\frac{6}{7}$ C. $\frac{2}{7}$ D. $\frac{4}{7}$ E. $\frac{5}{7}$

13. ($\star$) A teenage boy wrote his own age after his father's. From this new four digit number he subtracted the absolute value of the difference of their ages to get 4,289. Find the sum of their ages.

    A. 48 B. 52 C. 56 D. 59 E. 64

14. Consider the following:
    - On the 1^{st} day after Christmas, John gives his true love one present.
    - On the 2^{nd} day after Christmas, John gives his true love three additional presents $(1+2)$.
    - On the 3^{rd} day after Christmas, John gives his true love six additional presents $(1+2+3)$.
    - In general, on the k^{th} day after Christmas, John gives his true love $(1+2+\cdots+k)$ additional presents.

    How many presents in total has John given to his true love after 30 days of giving?

    A. 3654 B. 4060 C. 4495 D. 4960 E. 5456

15. Let $\vec{a}$ and $\vec{b}$ be two vectors. Suppose $|\vec{a}| = 3$ and $|\vec{b}| = 4$ and $|\vec{a}-\vec{b}| = \sqrt{13}$. Find the degree measurement of the angle between these two vectors. Assume the angle is between 0^o and 180^o.

    A. 135 B. 120 C. 30 D. 45 E. 60

16. How many real number solutions does the following equation have? $\sqrt{1-\sqrt{1+x}} = x$.

    A. 1 B. 2 C. 3 D. 4 E. None

17. ($\star$) A box contains red and blue marbles. The number of blue marbles is four times the number of red marbles. A marble will be randomly selected.

    - If the selected marble is red, then it will be returned into the box along with 15 additional blue marbles.
    - If the selected marble is blue, then it will be returned into the box; no additional marbles will be placed into the box.

    A second marble will then be selected. Prior to selecting the first marble, the probability of the second selected marble being blue equals 81%. How many blue marbles were originally in the box?

    A. 4 B. 16 C. 36 D. 56 E. 64

18. Determine the rational number represented by $2^{-3\log_2 5}$.

A. $\frac{1}{15}$ B. $\frac{1}{125}$ C. 0 D. -15 E. -125

19. Let y and z be negative real numbers satisfying $\frac{1}{y}+\frac{1}{z}-\frac{1}{y-z}=0$. Calculate $\frac{z}{y}$.

A. $\frac{1+\sqrt{5}}{2}$ B. $\frac{1-\sqrt{5}}{2}$ C. $\frac{-1+\sqrt{5}}{2}$ D. $\frac{-1-\sqrt{5}}{2}$ E. $\frac{\sqrt{5}}{2}$

20. The number 2^{29} is nine-digits long, and all nine digits are distinct. Which of the digits from 0 to 9 is missing?

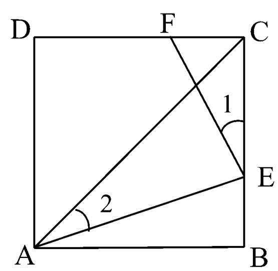
A. 3 B. 4 C. 6 D. 7 E. 9

21. Find the 2018^{th} smallest x, with $x > 1$ that satisfies the equation

$$\sin(\ln x) + 2\cos(505 \ln x)\sin(504 \ln x) = 0.$$

A. $e^{2018\pi}$ B. $e^{2\pi}$ C. 2018π D. e^{π} E. $e^{4\pi}$

22. ($\star$) For a given square $ABCD$, F is a point on $\overline{CD}$ satisfying $\overline{CF} = \frac{1}{3}\overline{CD}$, and E is a point on $\overline{BC}$ satisfying $\overline{BE} = \frac{1}{3}\overline{BC}$. Which of the following is true?



A. $\angle CAE = \angle CEF$ B. $\angle CAE > \angle CEF$ C. $\angle CAE < \angle CEF$ D. $\angle CAE = \frac{1}{2}\angle CEF$
E. None of the above is correct.

23. A pair of fair dice is rolled. What is the probability the first die is a 3 given the sum is a 6?

A. $\frac{1}{12}$ B. $\frac{1}{6}$ C. $\frac{1}{4}$ D. $\frac{1}{5}$ E. $\frac{2}{5}$

24. If the first day of March fell on a Tuesday one year, on what day of the week would August $15th$ fall on that same year?

A. Saturday B. Sunday C. Monday D. Tuesday E. Wednesday

25. Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} x & -1 \\ y & 1 \end{bmatrix}$. Find the values of real numbers x and y that satisfy $AB = BA$.

A. $x = \frac{5}{2}, y = \frac{-3}{2}$ B. $x = \frac{-3}{2}, y = \frac{-3}{2}$ C. $x = \frac{-3}{2}, y = \frac{5}{2}$ D. $x = \frac{5}{2}, y = \frac{3}{2}$
E. None of these

26. Assume $\frac{1}{a} - |a| = 1$ for a real number a. Find out the value of $\frac{1}{a} + |a|$.

A. $\frac{\sqrt{5}}{2}$ B. $\frac{-\sqrt{5}}{2}$ C. $\sqrt{5}$ D. $-\sqrt{5}$ E. 0

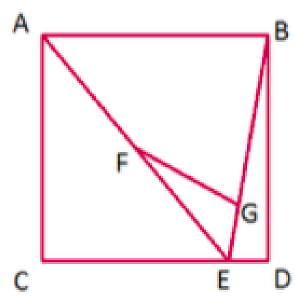
27. A container is filled with 8 liters of a 20% salt solution. How many liters of pure water must be added to produce a 15% salt solution?

A. $\frac{8}{3}$ B. $\frac{3}{5}$ C. $\frac{3}{8}$ D. 2 E. $\frac{13}{5}$

28. The polynomial $p(n) = n^2 + n + 41$ has a curious property: for small integer values of n, $p(n)$ is prime. For example, $p(1) = 43$, which is prime, $p(2) = 47$, which is also prime, and $p(3) = 53$, which is once again prime. Find the smallest positive integer n such that $p(n)$ is *not* prime.

A. 37 B. 38 C. 39 D. 40 E. 41

29. Assume the side length of square ABCD is 5. E is an arbitrary point on CD. F is the mid point of AE. G is a point on BE satisfying $BG = 3EG$. What is the area of triangle EFG?



A. 5/4 B. 5/6 C. 25/16 D. 3/2

30. Write $\frac{5-2i}{-3+i}$ in the form $a + bi$.

A. $\frac{-13}{10} + \frac{1}{10}i$ B. $\frac{-13}{8} - \frac{1}{8}i$ C. $\frac{-17}{8} + \frac{1}{8}i$ D. $\frac{-17}{10} + \frac{-1}{10}i$ E. $\frac{-17}{10} + \frac{1}{10}i$

31. Find the value of $\sqrt[3]{\sqrt{5}+2} - \sqrt[3]{\sqrt{5}-2}$.

A. -1 B. $2\sqrt[3]{2}$ C. $\sqrt[3]{2} - \sqrt[3]{2}$ D. 1 E. None of these

32. A right-truncated prime number is a prime number that yields another prime number when you delete it's ones digit. For example 431 is a right-truncated prime because 43 is prime. How many two digit right-truncated primes are there?

A. 8 B. 9 C. 10 D. 12 E. 13

33. Our codebreakers have recently uncovered a top secret organization of spies who collaborate on nefarious plots. Their activities are still mostly hidden from us, but we have managed to uncover the following facts:

- Any two distinct plots involve exactly one spy in common.
- Every spy is involved in exactly two plots.
- There are exactly four plots.

Exactly how many spies are there?

A. 5 B. 6 C. 8 D. 12 E. Infinitely many

34. ($\star$) How many integers between 50 and 500 are divisible by 11?

A. 40 B. 41 C. 39 D. 42 E. 43

35. Let x, y, and z be relatively prime integers. Which of the following expressions is equal to $\mathrm{lcm}(x^{12}, y^{20}, z^{16})$?

A. $(xyz)^{\gcd(12,20,16)}$ B. $(xyz)^{\mathrm{lcm}(12,20,16)}$ C. $(xyz)^{\mathrm{lcm}(12,20,16)\gcd(12,20,16)}$
D. $x^{3\gcd(12,20,16)}y^{5\gcd(12,20,16)}z^{4\gcd(12,20,16)}$ E. $x^{3\mathrm{lcm}(12,20,16)}y^{5\mathrm{lcm}(12,20,16)}z^{4\mathrm{lcm}(12,20,16)}$