2017 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 12, 13, 16, 24, 35

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Find the inverse function of $y = 3^{x+2}$.

   A. $y = \log_3 x - 2$ B. $y = \log_3 x + 2$ C. $y = \dfrac{x-9}{3}$ D. $y = \log_3 2x$ E. $y = 3x - 2$

> **Solution:**
> **A.** $y = \log_3 x - 2$
>
> We have $\log_3 y = x + 2$, so $x = \log_3 y - 2$. Hence, $f^{-1}(x) = \log_3 x - 2$.

2. Joe has a collection of nickels and dimes that is worth \$8.55. If the number of dimes was doubled and the number of nickels was increased by 18, the value of the coins would be \$15.25. How many dimes does he have?

   A. 58 B. 29 C. 18 D. 55 E. 62

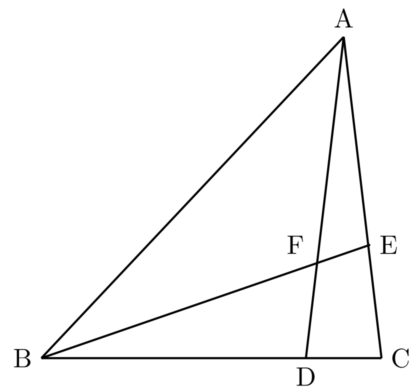
> **Solution: A.** 58
> Let x denote the number of nickels and y the number of dimes. Then
> $$5x + 10y = 855, \text{ which gives } x + 2y = 171.$$
> Then
> $$5(x + 18) + 10(2y) = 1525, \text{ giving } x + 18 + 4y = 305.$$
> Simplifying the second equation and multiplying the first by -1 yields a system of equations:
> $$\begin{aligned} -x - 2y &= -171 \\ x + 4y &= 287. \end{aligned}$$
> So $2y = 116$, and $y = 58$. Thus there are 58 dimes.

3. Let $A = \{10, 30, 50, 70, 90\}$, $B = \{10, 20, 30, 40, 50\}$, and $C = \{10, 20, 40, 60, 80, 90\}$ be sets with $U = \{10, 20, 30, \ldots, 100\}$ being the universal set. Which set below is the set $(A \cap B) \cap \overline{C}$? Note $\overline{C}$ is C complement.

   A. $\{10, 30, 50\}$ B. $\{10, 20, 30, 50, 70, 90\}$ C. $\{30, 50\}$ D. $\{50, 70\}$ E. $\{10\}$

> **Solution:**
> **C.** $\{30, 50\}$
>
> The union of A and B is $A \cap B = \{10, 30, 50\}$.
>
> The complement of C is $\overline{C} = \{30, 50, 70, 100\}$.
>
> So we have
> $$(A \cap B) \cap \overline{C} = \{30, 50\}.$$

4. The triangle ABC is shown below. D is a point on BC satisfying $BD : DC = 3 : 1$. E is a point on AC satisfying $AE : EC = 2 : 1$. F is the intersection of BE and AD. Which is ratio of the Area of AFB, Area of AFC, and the area of BFC (i.e $A_{AFB} : A_{AFC} : A_{BFC}$)?



A. $6:3:4$ B. $6:2:3$ C. $5:2:2$ D. $8:3:4$ E. $8:2:2$

Solution: B. $A_{AFB} : A_{AFC} : A_{BFC} = 6:2:3$

Since $AE : EC = 2 : 1$ we have Area (ABE) = 2 Area (BEC) and Area (AEF)= 2 Area(FEC). Subtracting the area of AEF from the area of ABE and subtracting the area of FEC from the area of BEC, we see Area (ABF) = 2 Area(BFC).

Similarly, since $BD : DC = 3 : 1$, we have Area (ABD) =3 Area (ADC) and Area (BFD) = 3Area (FDC). Therefore Area (ABF)= 3 Area(AFC). Therefore (b) is the right answer.

5. Consider any positive two-digit number ab. If this number is added to the number formed by reversing its digits, ba, what two integers greater than 1 will always divide their sum?

A. 11 and ab B. 11 and $a-b$ C. 11 and $a+b$ D. 22 and ab E. 22 and $a+b$

Solution: C. 11 **and** $a+b$

We want to find the divisors of $ab+ba$. We have

$$\begin{aligned} ab+ba &= 10a+b+10b+a \\ &= 10(a+b)+(a+b) \\ &= 11(a+b). \end{aligned}$$

6. Alex, Bob, Carm, and Dan will be asked a question about a recent robbery.

   i Alex or Bob or both will tell the truth.

   ii Carm or Dan or both will lie.

   iii If Alex lies then Carm lies as well.

   Which one of the following is possible.

   A. Only Dan will tell the truth. B. Only Carm will tell the truth. C. Only Bob and Carm will tell the truth. D. Only Carm will lie. E. Only Bob will lie.

Solution: D. Only Carm will lie

A. Only Dan will tell the truth. In this case both Alex and Bob lie, violating i.

B. Only Carm will tell the truth. In this case both Alex and Bob lie, violating i.

C. Only Bob and Carm will tell the truth. This means Alex has lied. By iii, this means Carm lies as well, which is not true.

D. Only Carm will lie. This means Alex and Bob told the truth- so i holds. Since Carm has lied, ii holds. Since Alex told the truth, iii. is true no matter what Carm does. So all three conditions hold. So D. is our answer.

E. Only Bob will lie. In this case, both Carm and Dan tell the truth, which violates ii.

7. A permutation is a rearrangement in letters in a string of text. For example: given the phrase "TomMarvoloRiddle", one of many permutations is"IamLordVoldemort". If you are given the string PPPSSU, how many ways can we arrange it?

A. 6 B. 72 C. 60 D. 180 E. 720

Solution: C. 60

This is a permutation. If we have n letters and k_1, k_2, $k_3, \ldots k_r$ repeated letters we have

$$\frac{n!}{k_1!k_2!\cdots k_3!}$$

ways to arrange the letters, where $m! = m(m-1)(m-2)\cdots(3)(2)(1)$. So to arrange PPPSSU we have

$$\frac{6!}{3!2!} = \frac{6\times5\times4\times3\times2\times1}{(3\times2\times1)(2\times1)} = 60$$

8. Consider the equation $x^2 + bx = -1$. Let b be determined by rolling a single six-sided fair die. The value of b is the number of dots on the side of the die facing upwards. What is the probability the equation has two distinct real roots?

A. 5/6 B. 1 C. 1/2 D. 1/3 E. 2/3

Solution:
E. $\frac{2}{3}$

We have $x^2 + bx + 1 = 0$, which has solutions

$$x = \frac{-b \pm \sqrt{b^2-4}}{2}$$

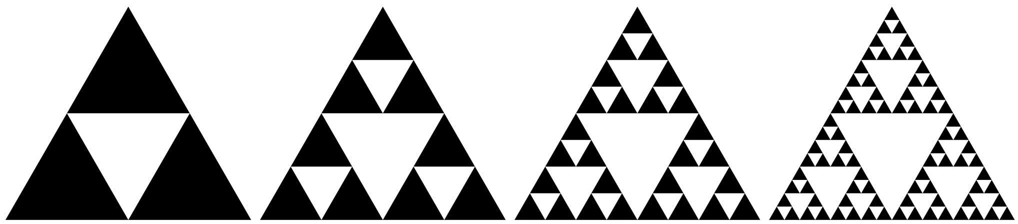
When $b = 1$, we have imaginary solutions. When $b = 2$, we have only one solution of $x = -1$. When $b \geq 3$, we see that the discriminant is larger than 0, so there will be two distinct roots.

So the probability is $\frac{2}{3}$.

9. Consider creating a sequence of triangles in the following manner

   1. Start with an equilateral triangle colored black.
   2. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle.
   3. Repeat step 2 with each of the remaining smaller triangles.

   Below are the first four iterations of the processes.



   In iteration 2, there are 9 black triangles, how many black triangles will there be after 7 iterations.

   A. 729 B. 1962 C. 2187 D. 4371 E. 6561

   Solution: C. 2187

   These are the Sierpinski triangle. It is known that on the k^{th} iteration, there are 3^k black rectangles. So there are $3^7 = 2187$.

   Another way would be notice that the first iteration has 3, the second has 9, and the third has 27. Each time we are taking each black triangle and turning it into 4 triangles, of which one is white and three are black. So the number of black triangles is being multiplied by a factor of three giving the above formula.

10. An observer measures the angle of elevation of the top of a tree to be $60°$ above horizontal. He then moves a distance of 10 feet further away from the tree (on level ground) and observes that the angle of elevation has decreases to $45°$. How tall (in feet) is the tree?

    A. $\dfrac{\sqrt{3}-1}{\sqrt{3}}$ B. $\dfrac{\sqrt{3}}{\sqrt{3}-1}$ C. $\dfrac{10\sqrt{3}}{\sqrt{3}-1}$ D. $\dfrac{\sqrt{3}-1}{10\sqrt{3}}$ E. $\dfrac{\sqrt{2}}{\sqrt{3}-1}$

    Solution: C. $\dfrac{10\sqrt{3}}{\sqrt{3}-1}$

    Let y denote the height of the tree and let x denote the original distance between the observer and the tree. Then $\tan 60° = \dfrac{y}{x} = \sqrt{3}$ and $\tan 45° = \dfrac{y}{x+10} = 1$. Solving for x in the first equation and substituting into the second, we have $\dfrac{y}{\frac{y}{\sqrt{3}}+10} = 1$. Thus $y = \dfrac{y}{\sqrt{3}} + 10$, so $y - \dfrac{y}{\sqrt{3}} = 10$, and so $y\left(1 - \dfrac{1}{\sqrt{3}}\right) = 10$. Hence $y = \dfrac{10}{1-1/\sqrt{3}} = \dfrac{10\sqrt{3}}{\sqrt{3}-1}$.

11. Find the sum of the first 30 terms of an arithmetic sequence in which the second term is -12 and the tenth term is 76.

    A. 5076 B. 4095 C. 4028 D. 3490 E. 1250

Solution: B. 4095

Let x denote the value of the first term in the sequence and let a denote the difference between each term and the previous. Since the tenth term is 76, counting from the second term we must have $-12 + 8a = 76$. Thus, $8a = 88$, and $a = 11$. So $x = -12 - a = -12 - 11 = -23$.

Now the sum of the first 30 terms is $-23 + (-23 + a) + (-23 + 2a) + (-23 + 3a) + \cdots + (-23 + 29a) = (-23)(30) + a(1 + 2 + 3 + \cdots + 29) = -690 + 11(29(30))/2 = -690 + 165(29) = -690 + 4785 = 4095$.

12. ($\star$) How many triples (a, b, c) of real numbers satisfy the equations

$$ab = c, \quad ac = b, \quad \text{and} \quad bc = a?$$

A. 2 B. 4 C. 5 D. 6 E. 8

Solution: C. 5

We have a trivial solution of $(0, 0, 0)$.

Suppose all of a, b and c are non-zero. Then equations $ab = c$ and $b = ac$ imply that $a^2c = c$. So $a^2 = 1$. Thus $a = \{-1, 1\}$. By similarity we have $b \in \{-1, 1\}$ and $c \in \{-1, 1\}$.

In order to satisfy the three equations, we must have exactly two -1 or all variables must be 1. So there are 5 solutions

$$(0, 0, 0), (1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1).$$

13. ($\star$) For t and k real numbers, assume $x_1 = \sin t$ and $x_2 = \cos t$ are solutions to $2x^2 - kx + 1 = 0$. What is the value of $\sin^4 t + \cos^4 t$?

A. 1/2 B. 1/3 C. 7/9 D. 1 E. 3/2

Solution: A. 1/2

Using the quadratic formula on the equation $2x^2 - kx + 1 = 0$, we have solution $x = \dfrac{k \pm \sqrt{k^2 - 8}}{4}$.

Without loss of generality, we can call these solutions x_1 and x_2, as $\sin t, \cos t$ respectively.

Thus we have

$$x_1 + x_2 = \frac{k + \sqrt{k^2 - 8}}{4} + \frac{k - \sqrt{k^2 - 8}}{4} = \frac{2k}{4} = \frac{k}{2}$$

and

$$x_1 \cdot x_2 = \frac{k + \sqrt{k^2 - 8}}{4} \cdot \frac{k - \sqrt{k^2 - 8}}{4} = \frac{k^2 - (k^2 - 8)}{16} = \frac{1}{2}.$$

Since $(\sin t + \cos t)^2 = 1 + 2 \sin t \cos t$, we have $\left(\dfrac{k}{2}\right)^2 = 1 + 2 \cdot \dfrac{1}{2} = 2$. Thus

$$(\sin t + \cos t)^4 = \left(\frac{k}{2}\right)^4 = 4.$$

Notice

$$\begin{aligned}(\sin t+\cos t)^4 &= \sin^4 t+4\sin^3 t\cos t+6\sin^2 t\cos^2 t+4\sin t\cos^3 t+\cos^4 t\\ &= \sin^4 t+4\sin t\cos t(\sin^2 t+\cos^2 t)+6\sin^2 t\cos^2 t+\cos^4 t\\ &= \sin^4 t+4\sin t\cos t+6(\sin t\cos t)^2+\cos^4 t\end{aligned}$$

Therefore

$$\begin{aligned}\sin^4 t+\cos^4 t &= (\sin t+\cos t)^4-4\sin t\cos t-6(\sin t\cos t)^2\\ &= 4-4\cdot\frac{1}{2}-6\left(\frac{1}{2}\right)^2=1/2.\end{aligned}$$

The answer is (a).

14. What is an equivalent expression for $\dfrac{3+i}{1-2i}$?

A. $\frac{1}{5}-\frac{7}{5}i$ B. $\frac{-6}{5}$ C. $\frac{1}{5}+\frac{7}{5}i$ D. $\frac{1}{5}-3i$ E. $\frac{1}{5}+3i$

Solution: C. $\frac{1}{5}+\frac{7}{5}i$

We have $\dfrac{3+i}{1-2i}=\dfrac{3+i}{1-2i}\dfrac{1+2i}{1+2i}=\dfrac{3+7i+2i^2}{1-4i^2}=\dfrac{3+7i-2}{1+4}=\dfrac{1+7i}{5}=\dfrac{1}{5}+\dfrac{7i}{5}$.

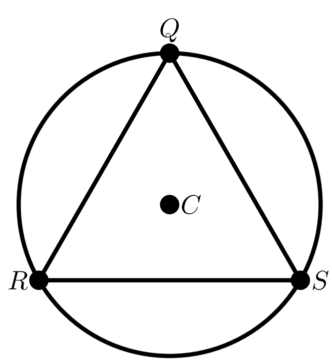
15. How many distinct ordered arrangements of the five letters *abcde* are there in which the first character is a, b, or c and the last character is c, d, or e?

A. 48 B. 36 C. 24 D. 12 E. 6

Solution: A. 48

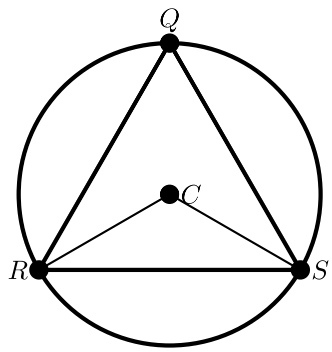
First we count the number of ways that a permutation could start with a or b and end with c, d, or e. So there are 2 choices for the first position, 3 choices for the last position, and 3! ways to arrange whatever is left for the middle three positions. That gives $2\cdot 3\cdot 3!=36$ ways. Next we'll count the number of ways for a permutation to start with c and end with d or e. This would be $1\cdot 2\cdot 3!=12$ ways. Together that gives 48 total permutations.

16. ($\star$) In the circle below with center C, $\overline{RS}=4$ and $\angle RQS=45°$. What is the circumference of the circle?



A. 4π B. $4\pi\sqrt{2}$ C. 8π D. $8\pi\sqrt{2}$ E. 12π

Solution: B. $4\pi\sqrt{2}$
Drawing $\overline{CR}$ and $\overline{CS}$, $\angle RCS$ is subtended by the same arc as $\angle RQS$, so $\angle RCS = 90^\circ$. Additionally, since $\overline{CR}$ and $\overline{CS}$ are both radii, they are equal, which means that triangle RCS is a $45-45-90$ triangle. Using the Pythagorean Theorem, we find that the side lengths of triangle RCS are $2\sqrt{2}$, which is the radius of the circle. Thus, the circumference of the circle is $2\pi 2\sqrt{2} = 4\pi\sqrt{2}$.



17. Assume $\sin\left(x - \frac{\pi}{2}\right) = \frac{7\sqrt{2}}{10}$. Determine $\sin(x)$.

A. $\pm\frac{1}{5\sqrt{2}}$ B. $-\frac{7\sqrt{2}}{10}$ C. $\frac{7\sqrt{2}}{10}$ D. $\pm\frac{\sqrt{2}}{4}$ E. $\frac{1}{\sqrt{2}}$

Solution: A. $\pm\frac{1}{5\sqrt{2}}$

We have
$$\sin\left(x - \frac{\pi}{2}\right) = -\cos(x).$$
Thus
$$\cos(x) = -\frac{7\sqrt{2}}{10}.$$
Then
$$\sin^2(x) = 1 - \cos^2(x) = 1 - \left(-\frac{7\sqrt{2}}{10}\right)^2 = 1 - \frac{49}{50} = \frac{1}{50}.$$
Therefore,
$$\sin(x) = \pm\frac{1}{5\sqrt{2}}.$$

18. Let $\vec{a}$ and $\vec{b}$ be vectors and let k be a scalar. If $\vec{a} + 2\vec{b}$ and $(k+1)\vec{a} + k\vec{b}$ are parallel, what is the value of k?

A. 1 B. -2 C. 2 D. $\frac{1}{2}$ E. $-\frac{1}{2}$

Solution: B. -2

Since $\vec{a}+2\vec{b}$ and $(k+1)\vec{a}+k\vec{b}$ are parallel, we have $\vec{a}+2\vec{b}=m(k+1)\vec{a}+mk\vec{b}$, for some constant m. Thus, $1=m(k+1)$ and $2=mk$, giving $m=\dfrac{2}{k}=\dfrac{1}{k+1}$. Hence, $2k+2=k$ so $k=-2$.

19. Find the number of positive integers n that satisfy the following equation:

$$(n^2-3n-3)^{n^2+2017}=(n^2-3n-3)^{92n+1}.$$

A. 1 B. 2 C. 3 D. 4 E. 5

Solution: C. 3

If $n^2-3n-3=1$, the above equation holds true since $1^{n^2+2017}=1^{92n+1}$ for all positive n. This implies that $n^2-3n-4=0$. Thus, $n=4$ or $n=-1$, giving one positive solution.

If $n^2-3n-3\neq 1$, then we must have $n^2+2017=92n+1$. This gives $n^2-92n+2016=0$, which is equivalent to $(n-36)(n-56)=0$. This yields two more positive solutions, $n=36$ or $n=56$.

Thus, we have three positive integer solutions for n.

20. In a list of 200 numbers, every number (except the first and last) is equal to the sum of the adjacent numbers in the list. For example the sequence *could* look like $3,7,4,-3,\ldots$ or $-2,5,7,2,\ldots$. The sum of the first 200 numbers is equal to the sum of the first 100 numbers. If the 48^{th} number in the list is 2017, what is the sum of all 200 numbers?

A. 4051 B. 2034 C. 0 D. -2034 E. -6051

Solution: E. -6051

Let x_n be the n^{th} term of the list. Suppose $x_1=a$ and $x_2=b$. Then since $x_2=x_1+x_3, x_3=b-a$. We can continue this way and we see $x_4=-a, x_5=-b, x_6=a-b, x_7=a, x_8=b,\ldots$. So we see the sequence will repeat every six terms.

Let S_n be the sum of the first n term. Then $S_6=a+b+(b-a)-a-b+(a-b)=0$. So, $S_{100}=S_4=2b-a$ and $S_{200}=S_2=a+b$.

Since $S_{100}=S_{200}, 2b-a=a+b$. Thus $b=2a$.

Since $x_{48}=2017$ and $x_{48}=a-b=a-2a=2017$. Thus $a=-2017$. And $b=-4034$. Thus $S_{200}=-2017-4034=-6051$

21. Two tennis players will play a series of matches until one player wins 4 matches. Assume that the players are of equal ability, i.e. each player has a 50/50 chance of winning any match. Determine the probability that exactly 6 matches will be needed to determine the champion.

A. 1/64 B. 1/32 C. 5/64 D. 5/32 E. 5/16

Solution: E. 5/16

Consider Player A, who can win after 6 matches in the following ways:

LLWWWW LWLWWW LWWLWW LWWWLW WLLWWW
WLWLWW WLWWLW WWLLWW WWLWLW WWWLLW.

Each of these outcomes has a probability of $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$.

So player A has a probability of

$$10\left(\frac{1}{64}\right) = \frac{5}{32}$$

of being a champion.

Since the players are of equal ability, Player B also has a $\frac{5}{32}$ probability of winning after 6 matches.

Since Player A winning and Player B winning are mutually exclusive the probability 6 matches determine the winner is

$$\frac{5}{32} + \frac{5}{32} = \frac{5}{16}$$

22. In a triangle $\triangle DEF$, $\overline{DE} = \sqrt{2017+2018}$, $\overline{EF} = 2017$, and $\overline{DF} = 2018$.
Calculate $\sin(\angle D) \cdot \cos(\angle F)$.

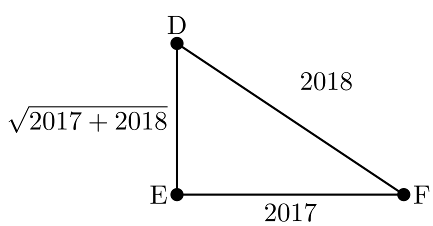
A. 1 B. $\frac{2017}{2018}$ C. $\frac{\sqrt{2017}}{\sqrt{2018}}$ D. $\frac{2017^2}{2018^2}$ E. $\frac{2018}{2017}$

Solution: D. $\frac{2017^2}{2018^2}$

We have $\overline{DE} = \sqrt{(2017+2018)\cdot 1} = \sqrt{(2017+2018)(2018-2017)} = \sqrt{2018^2 - 2017^2}$.

Therefore, $\overline{DE}^2 = 2018^2 - 2017^2 = \overline{DF}^2 - \overline{EF}^2$. Hence, $\overline{DF}^2 = \overline{DE}^2 + \overline{EF}^2$. This implies that $\angle E = 90^\circ$.

Now $\sin(\angle D) \cdot \cos(\angle F) = \frac{2017}{2018} \cdot \frac{2017}{2018} = \frac{2017^2}{2018^2}$.



23. What is the value of the positive integer n for which the least common multiple of 36 and n is 500 greater than the greatest common divisor of 36 and n?

A. 48 B. 80 C. 96 D. 112 E. None of these

Solution: E. None of these

The least common multiple must be even since it is divisible by 36. Thus, the greatest common divisor must be even since it is a difference of two even numbers, the LCM and 500.

The even divisors of 36 are: $2, 4, 6, 12, 18, 36$.

If we add 500 to each, we get: $502, 504, 506, 512, 518, 536$

Only 504 is divisible by 36. We have, $504 = 36 \cdot 14 = 2^3 \cdot 3^2 \cdot 7$.

Then $\operatorname{lcm}(2^2 3^2, n) = 2^3 3^2 7$ and $\gcd(2^2 3^2, n) = 2^2$.

So $n = 2^3 \cdot 7 = 56$, which is not one of our options.

24. ($\star$) If $a \cdot b \neq 1$, $8a^2 + 2017a + 9 = 0$ and $9b^2 + 2017b + 8 = 0$. What is the value of $\frac{a}{b}$?

A. $\frac{\sqrt{2}}{3}$ B. $\frac{9}{8}$ C. $\frac{2017}{72}$ D. $\frac{72}{2017}$ E. $\frac{8}{9}$

Solution: B. 9/8

We want to look at the solutions to $8x^2 + 2017x + 9 = 0$. We see that a will be a solution. Dividing the second equation by b^2, we have $8\left(\frac{1}{b}\right)^2 + 2017\left(\frac{1}{b}\right) + 9 = 0$. Since $a \cdot b \neq 1$, $\frac{1}{b}$ is a second solution, distinct from a. Thus there are the two distinct solutions of $8x^2 + 2017x + 9 = 0$.

Letting $8x^2 + 2017x + 9 = ax^2 + bx + c$ and using the quadratic formula we have solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Multiplying these two solutions, we have

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{(-b)^2 - (b^2 - 4ac}{(2a)^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Since our solutions are a and $\frac{1}{b}$, we have $a \cdot \frac{1}{b} = 9/8$. So (b) is the correct answer.

25. What is $\displaystyle\sum_{k=1}^{5} \cos^2(k\pi/2)$?

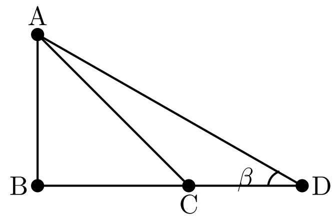
A. -1 B. 0 C. 1 D. 2 E. 3

Solution: D. 2

We have

$$\begin{aligned}\sum_{k=1}^{5} \cos^2(k\pi/2) &= \cos^2(\pi/2) + \cos^2\pi + \cos^2(3\pi/2) + \cos^2(2\pi) + \cos^2(5\pi/2) \\ &= 0 + (-1)^2 + 0 + 1^2 + 0 = 2.\end{aligned}$$

26. In isosceles triangle $\triangle ABC$, $\angle B = 90°$. Let D be a point on the extension of the line BC satisfying $\overline{AC} = \overline{CD}$. If $\angle ADC = \beta$, find the value of $\cos^2 \beta$.



A. $2\sqrt{2}$ B. $\frac{2-\sqrt{2}}{4}$ C. $\frac{\sqrt{2}}{2}$ D. 2 E. $\frac{\sqrt{2}+2}{4}$

Solution: E. $\frac{\sqrt{2}+2}{4}$

In $\triangle ACD$, since $\overline{AC} = \overline{CD}$, $\angle CAD = \angle ADC$. Therefore, $\angle ACB = 2\angle ADC = 2\beta$.

In $\triangle ACB$, since $\overline{AB} = \overline{CB}$, $\angle ACB = 45°$. Thus, $\cos^2 \beta = \frac{\cos 2\beta + 1}{2} = \frac{\cos 45° + 1}{2} = \frac{\frac{\sqrt{2}}{2} + 1}{2} = \frac{\sqrt{2}+2}{4}$.

27. Initially, there are three plants on the windowsill of the bedroom of Margo, Edith, and Agnes. From left to right they are a rose, violet and tulip. Every morning, Margo waters the plants and changes the plant on the left with the one in the middle. Every afternoon, Agnes waters them and changes the plant on the right with the one in the center. After one year (365 days), at the end of the day, what is the order of the plants?

A. rose, tulip, violet B. violet, rose, tulip C. violet, rose, tulip D. tulip, violet, rose E. tulip, rose, violet

Solution: E. tulip, rose, violet

Let R be the rose, V the violet, and T the tulip. We have:

Start	RVT
Day 1	VRT VTR
Day 2	TVR TRV
Day 3	RTV RVT

So at the end of every 3 days, we are back to the starting position, RVT. The question wants to know what the order is at the **end** of day 365, which is the same as the end of day 2. So TRV, tulip, rose, violet.

28. Find the value of c so that the equation $x^2 - 6x + c = 0$ has only one real solution.

A. -6 B. -3 C. 0 D. 6 E. 9

Solution: E. 9

Solution A: Completing the square we have

$$y = x^2 - 6x + 9 + (-9 + c) = (x - 3)^2 + (-9 + c).$$

The vertex is $(-3, -9 + c)$. To only have one intersection with the x-axis, we need $-9 + c = 0$. So $c = 9$.

Solution B: Suppose $x^2 - 6x + c = 0$ has only one real solution, $x = a$. Then $x^2 - 6x + c = (x - a)^2$. Expanding the right-hand-side yields $2a = 6$ and $a^2 = c$, hence $a = 3$, so $c = 9$.

29. A large container, labeled R, is partially filled with 4 quarts of red paint. Another large container, labeled W, is partially filled with 5 quarts of white paint. A small empty bottle is completely filled with red paint taken from R, and the contents of the bottle are then emptied into W. After thorough mixing of the contents of W, the bottle is completely filled with some of this mixture from W, and the contents of the bottle then emptied into R. The ratio of red paint to white in R is now 3:1. What is the size of the bottle, in quarts?

A. $5/3$ B. $2\sqrt{5}3$ C. $5/4$ D. $\sqrt{5}/2$ E. $3/2$

Solution: C. $5/4$

Let x denote the size of the bottle in quarts.

After the first transfer R has $(4 - x)$ red and W has 5 white and x red. So the ratio of red in container W is $\dfrac{x}{5+x}$ and the ratio of white is $\dfrac{5}{5+x}$.

When performing the second transfer, the amount of red in the transfer bottle is $\dfrac{x^2}{5+x}$ and the amount of white is $\dfrac{5x}{5+x}$.

So, after the second transfer R has $\left(4 - x + \dfrac{x^2}{5+x}\right)$ red and $\dfrac{5x}{5+x}$ white.

So the ratio is R of red to white is

$$\frac{4 - x + \frac{x^2}{5+x}}{\frac{5x}{5+x}} = \frac{20 - x}{5x} = \frac{3}{1}.$$

Solving we have $x = \dfrac{5}{4}$

30. Simplify $37.5\% \times \dfrac{600}{75} \times 8\dfrac{1}{3}\%$.

A. $\dfrac{1}{4}$ B. $\dfrac{1}{5}$ C. $\dfrac{1}{8}$ D. $\dfrac{1}{15}$ E. $\dfrac{1}{3}$

Solution:

A. $\frac{1}{4}$

We can express $37.5\% \times \frac{600}{75} \times 8\frac{1}{3}\%$ as

$$\frac{375}{1000} \times \frac{600}{75} \times \frac{8+1/3}{100} = \frac{375}{1000} \times \frac{600}{75} \times \frac{25}{300}.$$

Canceling terms, this becomes

$$\frac{375}{1000} \times \frac{2}{3} = \frac{375}{1500} = \frac{15}{60} = \frac{1}{4}.$$

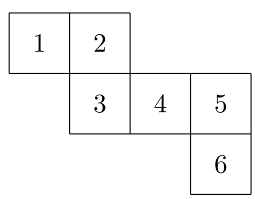
31. Let $x = 1202_3$ and $y = 221_3$ represent numbers in a base three number system. What is the product xy_3 in a base three number system.

A. 1121112 B. 1220202 C. 2021010 D. 2121010 E. None of these

Solution: A. 1121112

First we translate to base 10, so $x_{10} = 1(3^3) + 2(3^2) + 0(3^1) + 2(3^0) = 47$ and $y_{10} = 2(3^2) + 2(3^1) + 1(3^0) = 25$. Now $x_{10}y_{10} = 1175$. Translating back to base three, we have $1(3^6) + 1(3^5) + 2(3^4) + 1(3^3) + 1(3^2) + 1(3^1) + 2(3^0) = 1121112$.

32. The following net can be folded into a cube.



What is the number on the face that is on the opposite side of "6"?

A. 2 B. 1 C. 5 D. 3 E. 4

Solution: A. 2

We see that as we fold in the squares starting with with 4 on the bottom, first the 3 and 5 will be raised up on opposite sides. Then the 2 and 6 will be folded in on opposite sides, with 1 on the top.

33. Luke earns \$4 more in four hours than Manny earns in 3 hours. And Manny earns \$0.50 more in four hours than Luke earns in 5 hours. How much does Luke make per hour?

A. \$15.00 B. \$15.50 C. \$16.25 D. \$17.50 E. \$19.00

Solution: D. 17.50

Let l be the amount Luke earns per hour and m be the amount Manny makes per hour. So $4l = 3m+4$ and $4m = 5l + 0.5$.

Now

$$4l = 3m + 4 \Rightarrow \frac{4l-4}{3} = m$$

So

$$\begin{aligned} 4m = 5l + 0.5 &\Rightarrow \frac{16l}{3} - \frac{16}{3} - \frac{1}{2} = 5l \\ &\Rightarrow \frac{16l}{3} - \frac{35}{6} = 5l \Rightarrow \frac{16l}{15} - \frac{35}{30} = l \\ &\Rightarrow \frac{l}{15} = \frac{35}{30} \Rightarrow l = \frac{35}{2} = 17.50 \end{aligned}$$

Luke makes \$17.50 per hour.

34. Which of the following is a root of the polynomial $p(x) = x^3 - 5x^2 - 7x + 35$.

A. -5 B. $-\sqrt{7}$ C. $-\sqrt{5}$ D. 7 E. 35

Solution: B. $-\sqrt{7}$

By the Rational Root Theorem, the only possible integer roots are divisors of 35, meaning $\pm 1, \pm 5, \pm 7, \pm 35$. Moving left-to-right, the first candidate that actually is a root is $x = 5$. So now we can factor:

$$p(x) = (x-5)(x^2-7).$$

Thus the other two roots are $x = \pm\sqrt{7}$.

35. ($\star$) The number $2^{48} - 1$ is divisible by two numbers between 60 and 70. Find these two numbers.

A. $61, 63$ B. $61, 65$ C. $63, 65$ D. $63, 67$ E. $67, 69$

Solution: C. $63, 65$

We have

$$\begin{aligned} (2^{48}-1) &= (2^{24}-1)(2^{24}+1) \\ &= (2^{12}-1)(2^{12}+1)(2^{24}+1) \\ &= (2^{6}-1)(2^{6}+1)(2^{12}+1)(2^{24}+1) \\ &= (63)(65)(2^{12}+1)(2^{24}+1). \end{aligned}$$