2017 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 12, 13, 16, 24, 35

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Find the inverse function of $y = 3^{x+2}$.

   A. $y = \log_3 x - 2$ B. $y = \log_3 x + 2$ C. $y = \dfrac{x-9}{3}$ D. $y = \log_3 2x$ E. $y = 3x - 2$

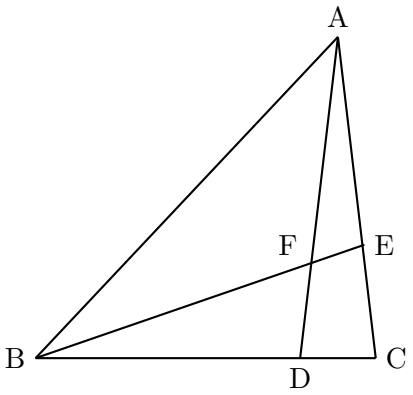
2. Joe has a collection of nickels and dimes that is worth \$8.55. If the number of dimes was doubled and the number of nickels was increased by 18, the value of the coins would be \$15.25. How many dimes does he have?

   A. 58 B. 29 C. 18 D. 55 E. 62

3. Let $A = \{10, 30, 50, 70, 90\}$, $B = \{10, 20, 30, 40, 50\}$, and $C = \{10, 20, 40, 60, 80, 90\}$ be sets with $U = \{10, 20, 30, \ldots, 100\}$ being the universal set. Which set below is the set $(A \cap B) \cap \overline{C}$? Note $\overline{C}$ is C complement.

   A. $\{10, 30, 50\}$ B. $\{10, 20, 30, 50, 70, 90\}$ C. $\{30, 50\}$ D. $\{50, 70\}$ E. $\{10\}$

4. The triangle ABC is shown below. D is a point on BC satisfying $BD : DC = 3 : 1$. E is a point on AC satisfying $AE : EC = 2 : 1$. F is the intersection of BE and AD. Which is ratio of the Area of AFB, Area of AFC, and the area of BFC (i.e $A_{AFB} : A_{AFC} : A_{BFC}$)?



   A. 6 : 3 : 4 B. 6 : 2 : 3 C. 5 : 2 : 2 D. 8 : 3 : 4 E. 8 : 2 : 2

5. Consider any positive two-digit number ab. If this number is added to the number formed by reversing its digits, ba, what two integers greater than 1 will always divide their sum?

   A. 11 and ab B. 11 and $a - b$ C. 11 and $a + b$ D. 22 and ab E. 22 and $a + b$

6. Alex, Bob, Carm, and Dan will be asked a question about a recent robbery.

   i Alex or Bob or both will tell the truth.

   ii Carm or Dan or both will lie.

   iii If Alex lies then Carm lies as well.

   Which one of the following is possible.

   A. Only Dan will tell the truth. B. Only Carm will tell the truth. C. Only Bob and Carm will tell the truth. D. Only Carm will lie. E. Only Bob will lie.

7. A permutation is a rearrangement in letters in a string of text. For example: given the phrase "TomMarvoloRiddle", one of many permutations is"IamLordVoldemort". If you are given the string PPPSSU, how many ways can we arrange it?

   A. 6 B. 72 C. 60 D. 180 E. 720

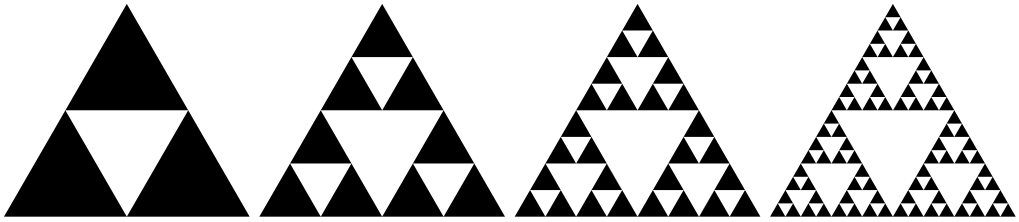
8. Consider the equation $x^2 + bx = -1$. Let b be determined by rolling a single six-sided fair die. The value of b is the number of dots on the side of the die facing upwards. What is the probability the equation has two distinct real roots?

   A. 5/6 B. 1 C. 1/2 D. 1/3 E. 2/3

9. Consider creating a sequence of triangles in the following manner

   1. Start with an equilateral triangle colored black.
   2. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle.
   3. Repeat step 2 with each of the remaining smaller triangles.

   Below are the first four iterations of the processes.



   In iteration 2, there are 9 black triangles, how many black triangles will there be after 7 iterations.

   A. 729 B. 1962 C. 2187 D. 4371 E. 6561

10. An observer measures the angle of elevation of the top of a tree to be 60° above horizontal. He then moves a distance of 10 feet further away from the tree (on level ground) and observes that the angle of elevation has decreases to 45°. How tall (in feet) is the tree?

    A. $\dfrac{\sqrt{3}-1}{\sqrt{3}}$ B. $\dfrac{\sqrt{3}}{\sqrt{3}-1}$ C. $\dfrac{10\sqrt{3}}{\sqrt{3}-1}$ D. $\dfrac{\sqrt{3}-1}{10\sqrt{3}}$ E. $\dfrac{\sqrt{2}}{\sqrt{3}-1}$

11. Find the sum of the first 30 terms of an arithmetic sequence in which the second term is -12 and the tenth term is 76.

    A. 5076 B. 4095 C. 4028 D. 3490 E. 1250

12. ($\star$) How many triples (a, b, c) of real numbers satisfy the equations

$$ab = c, \quad ac = b, \quad \text{and} \quad bc = a?$$

    A. 2 B. 4 C. 5 D. 6 E. 8

13. ($\star$) For t and k real numbers, assume $x_1 = \sin t$ and $x_2 = \cos t$ are solutions to $2x^2 - kx + 1 = 0$. What is the value of $\sin^4 t + \cos^4 t$?

    A. 1/2 B. 1/3 C. 7/9 D. 1 E. 3/2

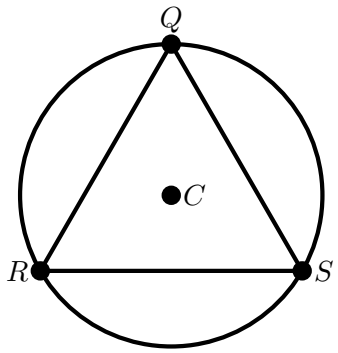
14. What is an equivalent expression for $\dfrac{3+i}{1-2i}$?

    A. $\dfrac{1}{5} - \dfrac{7}{5}i$ B. $\dfrac{-6}{5}$ C. $\dfrac{1}{5} + \dfrac{7}{5}i$ D. $\dfrac{1}{5} - 3i$ E. $\dfrac{1}{5} + 3i$

15. How many distinct ordered arrangements of the five letters *abcde* are there in which the first character is a, b, or c and the last character is c, d, or e?

    A. 48 B. 36 C. 24 D. 12 E. 6

16. ($\star$) In the circle below with center C, $\overline{RS} = 4$ and $\angle RQS = 45^\circ$. What is the circumference of the circle?



    A. 4π B. $4\pi\sqrt{2}$ C. 8π D. $8\pi\sqrt{2}$ E. 12π

17. Assume $\sin\left(x - \dfrac{\pi}{2}\right) = \dfrac{7\sqrt{2}}{10}$. Determine $\sin(x)$.

    A. $\pm\dfrac{1}{5\sqrt{2}}$ B. $-\dfrac{7\sqrt{2}}{10}$ C. $\dfrac{7\sqrt{2}}{10}$ D. $\pm\dfrac{\sqrt{2}}{4}$ E. $\dfrac{1}{\sqrt{2}}$

18. Let $\vec{a}$ and $\vec{b}$ be vectors and let k be a scalar. If $\vec{a} + 2\vec{b}$ and $(k+1)\vec{a} + k\vec{b}$ are parallel, what is the value of k?

    A. 1 B. -2 C. 2 D. $\dfrac{1}{2}$ E. $-\dfrac{1}{2}$

19. Find the number of positive integers n that satisfy the following equation:

$$(n^2 - 3n - 3)^{n^2+2017} = (n^2 - 3n - 3)^{92n+1}.$$

    A. 1 B. 2 C. 3 D. 4 E. 5

20. In a list of 200 numbers, every number (except the first and last) is equal to the sum of the adjacent numbers in the list. For example the sequence *could* look like $3, 7, 4, -3, \ldots$ or $-2, 5, 7, 2, \ldots$. The sum of the first 200 numbers is equal to the sum of the first 100 numbers. If the 48^{th} number in the list is 2017, what is the sum of all 200 numbers?

    A. 4051 B. 2034 C. 0 D. -2034 E. -6051

21. Two tennis players will play a series of matches until one player wins 4 matches. Assume that the players are of equal ability, i.e. each player has a 50/50 chance of winning any match. Determine the probability that exactly 6 matches will be needed to determine the champion.

A. 1/64 B. 1/32 C. 5/64 D. 5/32 E. 5/16

22. In a triangle $\triangle DEF$, $\overline{DE} = \sqrt{2017 + 2018}$, $\overline{EF} = 2017$, and $\overline{DF} = 2018$.
Calculate $\sin(\angle D) \cdot \cos(\angle F)$.

A. 1 B. $\dfrac{2017}{2018}$ C. $\dfrac{\sqrt{2017}}{\sqrt{2018}}$ D. $\dfrac{2017^2}{2018^2}$ E. $\dfrac{2018}{2017}$

23. What is the value of the positive integer n for which the least common multiple of 36 and n is 500 greater than the greatest common divisor of 36 and n?

A. 48 B. 80 C. 96 D. 112 E. None of these

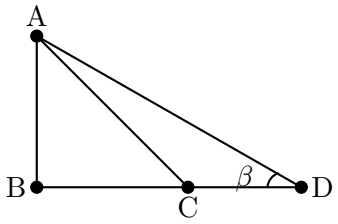
24. ($\star$) If $a \cdot b \neq 1$, $8a^2 + 2017a + 9 = 0$ and $9b^2 + 2017b + 8 = 0$. What is the value of $\dfrac{a}{b}$?

A. $\dfrac{\sqrt{2}}{3}$ B. $\dfrac{9}{8}$ C. $\dfrac{2017}{72}$ D. $\dfrac{72}{2017}$ E. $\dfrac{8}{9}$

25. What is $\displaystyle\sum_{k=1}^{5} \cos^2(k\pi/2)$?

A. -1 B. 0 C. 1 D. 2 E. 3

26. In isosceles triangle $\triangle ABC$, $\angle B = 90^\circ$. Let D be a point on the extension of the line BC satisfying $\overline{AC} = \overline{CD}$. If $\angle ADC = \beta$, find the value of $\cos^2\beta$.



A. $2\sqrt{2}$ B. $\dfrac{2-\sqrt{2}}{4}$ C. $\dfrac{\sqrt{2}}{2}$ D. 2 E. $\dfrac{\sqrt{2}+2}{4}$

27. Initially, there are three plants on the windowsill of the bedroom of Margo, Edith, and Agnes. From left to right they are a rose, violet and tulip. Every morning, Margo waters the plants and changes the plant on the left with the one in the middle. Every afternoon, Agnes waters them and changes the plant on the right with the one in the center. After one year (365 days), at the end of the day, what is the order of the plants?

A. rose, tulip, violet B. violet, rose, tulip C. violet, rose, tulip D. tulip, violet, rose
E. tulip, rose, violet

28. Find the value of c so that the equation $x^2 - 6x + c = 0$ has only one real solution.

A. -6 B. -3 C. 0 D. 6 E. 9

29. A large container, labeled R, is partially filled with 4 quarts of red paint. Another large container, labeled W, is partially filled with 5 quarts of white paint. A small empty bottle is completely filled with red paint taken from R, and the contents of the bottle are then emptied into W. After thorough mixing of the contents of W, the bottle is completely filled with some of this mixture from W, and the contents of the bottle then emptied into R. The ratio of red paint to white in R is now 3:1. What is the size of the bottle, in quarts?

A. 5/3 B. $2\sqrt{53}$ C. 5/4 D. $\sqrt{5}/2$ E. 3/2

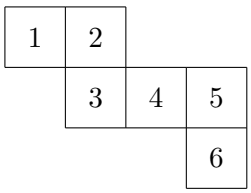
30. Simplify $37.5\% \times \dfrac{600}{75} \times 8\dfrac{1}{3}\%$.

A. $\dfrac{1}{4}$ B. $\dfrac{1}{5}$ C. $\dfrac{1}{8}$ D. $\dfrac{1}{15}$ E. $\dfrac{1}{3}$

31. Let $x = 1202_3$ and $y = 221_3$ represent numbers in a base three number system. What is the product xy_3 in a base three number system.

A. 1121112 B. 1220202 C. 2021010 D. 2121010 E. None of these

32. The following net can be folded into a cube.



What is the number on the face that is on the opposite side of "6"?

A. 2 B. 1 C. 5 D. 3 E. 4

33. Luke earns \$4 more in four hours than Manny earns in 3 hours. And Manny earns \$0.50 more in four hours than Luke earns in 5 hours. How much does Luke make per hour?

A. \$15.00 B. \$15.50 C. \$16.25 D. \$17.50 E. \$19.00

34. Which of the following is a root of the polynomial $p(x) = x^3 - 5x^2 - 7x + 35$.

A. -5 B. $-\sqrt{7}$ C. $-\sqrt{5}$ D. 7 E. 35

35. ($\star$) The number $2^{48} - 1$ is divisible by two numbers between 60 and 70. Find these two numbers.

A. $61, 63$ B. $61, 65$ C. $63, 65$ D. $63, 67$ E. $67, 69$