2016 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 12, 18, 22, 23, 29

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Three students were asked to add two positive numbers. The first student multiplied the two numbers and got 165. The second student subtracted the two numbers and got 28. If the third student was the only one who got the correct answer, what did her answer have to be?

   A. 33 B. 38 C. 5 D. 28 E. 17

> **Solution: B.** 38
>
> Let x and y denote the two numbers, where $x \geq y$. Then $xy = 165$ and $x - y = 28$. Thus, $x = y + 28$, and so $xy = (y+28)y = y^2 + 28y = 165$. Equivalently $y^2 + 28y - 165 = 0$. Factoring, we get $(y-5)(y+33) = 0$. Since $y > 0$, we must have $y = 5$, and then $x = 33$. Then the sum $x + y = 33 + 5 = 38$.

2. A girl who is 4 ft. tall is standing next to a telephone pole. At 2 PM her shadow is 7 ft. long and the telephone pole's shadow is 31.5 ft. long. How tall (in feet) is the telephone pole?

   A. 126 B. 55.125 C. 36 D. 28 E. 18

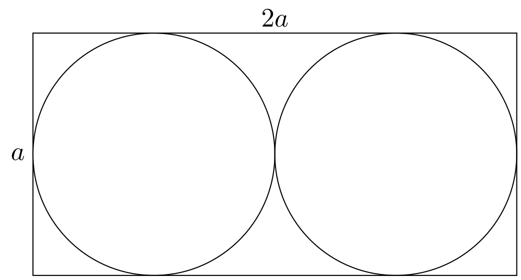
> **Solution: E.** 18
>
> Using similar triangles we have $\dfrac{x}{31.5} = \dfrac{4}{7}$, so $x = \dfrac{4(31.5)}{7} = 18$.

3. Consider four integers $a, b, a-b$, and $a+b$ that are all prime numbers. Which of the following is true about the sum of all four integers?

   A. divisible by 2 B. divisible by 3 C. divisible by 5 D. divisible by 7 E. none of these

> **Solution: E. none of these**
>
> Since $(a-b) + (a+b) = 2a$, which is even, we see that $a-b$ and $a+b$ must have the same parity (both be even or both be odd). The only even prime is 2, so they both must be odd. Thus, exactly one of a or b is even i.e. 2. Since $a + b > a - b > 2$, it follows that $b = 2$.
>
> Then $a-2, a, a+2$ are consecutive odd primes. Given 3 consecutive odd numbers, at least one must be divisible by 3. Since each is prime, we must have $a - 2 = 3$. So $a = 5$.
>
> Then we have
> $$a + b + (a-b) + (a+b) = 5 + 2 + (5-2) + (5+2) = 17.$$
>
> So the sum is prime, and since this prime number is greater than 7, it is not divisible by 2, 3, 5, or 7.

4. Consider two circles of equal radius tangent to each other inscribed in a rectangle with width a and length $2a$ as shown below.



Let X denote the total area of the two circles and Y denote the area of the rectangle that is **not** contained inside either circle. Find $\frac{X}{Y}$.

A. $\frac{\frac{\pi}{2}}{4-\pi}$ B. $\frac{\frac{3\pi}{4}}{4-\pi}$ C. $\frac{\pi}{4-\pi}$ D. $\frac{\frac{3\pi}{2}}{4-\pi}$ E. None of these.

Solution: C. $\frac{\pi}{4-\pi}$

The radius of each circle is $\frac{a}{2}$. So the area of one circle is $A_c = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$. Thus the area of both circles is

$$A_{2c} = \frac{\pi a^2}{2} = X.$$

The area inside the rectangle and outside the circles is $A_r - A_{2c}$, where $A_r = a \cdot 2a = 2a^2$. So

$$Y = 2a^2 - \frac{\pi a^2}{2} = \frac{4a^2 - \pi a^2}{2}.$$

Therefore,

$$\frac{X}{Y} = \frac{\frac{\pi a^2}{2}}{\frac{a^2(4-\pi)}{2}} = \frac{\pi}{4-\pi}.$$

5. We can express 2016 in the form $2^x(2^n - 1)$, for some n. What is x?

A. $2n-1$ B. $n-1$ C. $n-2$ D. $2n$ E. n^2

Solution: B. $n-1$

If we factor 2016, we have

$$2016 = 2^5 3^2 7.$$

Now $2^n - 1$ is always odd, so will not be divisible by 2. So $2^n - 1 = 3^2 \cdot 7 = 63$ and $2^x = 2^5$. We have

$$2^5(2^6 - 1) = 32(63) = 2016,$$

so $n = 6$ and $x = 5$. Thus $x = n - 1$

6. Consider

$$WELOVEMATHEMATICSWELOVEMATHEMATICS\ldots$$

If this pattern continues indefinitely, what will be the 203^{rd} letter in the pattern?

A. S B. W C. T D. C E. I

Solution: D. C

The repeated phrase has 17 letters, so we need to find the remainder when 203 is divided by 17, which is 16. The sixteenth letter is C.

7. The wheels on one bike have a radius of 1/2 ft. The wheels on a larger bike have a radius of 1 ft. If bikers using these two bikes both ride for one mile (5280 ft), how many more complete rotations will the smaller bike's wheels make than the wheels on the larger bike?

A. 840 B. 980 C. 1680 D. 2640 E. 16579

Solution: A. 840

The number of rotations for each wheel is $\frac{5280}{2\pi r}$. For the wheel of radius 1/2, the number of rotations is $\frac{5280}{\pi}$. For the wheel of radius 1 the number of rotations is $\frac{5280}{2\pi}$.

Therefore the difference in the number of full rotations is the whole number part of

$$\frac{5280}{\pi} - \frac{5280}{2\pi} = \frac{2(5280) - 5280}{2\pi} = \frac{5280}{2\pi},$$

which gives 840 more full rotations.

8. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We call a set of three numbers *acceptable* if it has at most one odd number. For example $\{1, 2, 6\}$ and $\{2, 4, 6\}$ are both acceptable. If we randomly choose three numbers from set A, what is the probability the set will be acceptable?

A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $\frac{1}{21}$ D. $\frac{5}{14}$ E. $\frac{17}{42}$

Solution: E. $\frac{17}{42}$

The total number of sets of size 3 is

$$\binom{9}{3} = \frac{9!}{3!6!} = 84.$$

The number of sets with no odd numbers is

$$\binom{4}{3} = \frac{4!}{3!1!} = 4.$$

The number of sets with exactly 1 odd number is

$$5\binom{4}{2} = 5 \cdot \frac{4!}{2!2!} = 30.$$

So the probability a set is acceptable is

$$\frac{34}{84} = \frac{17}{42}.$$

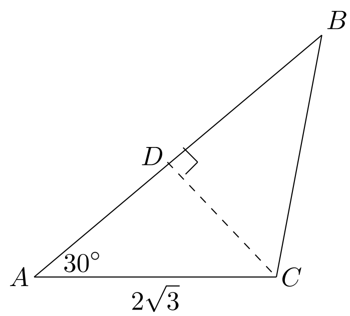
9. In triangle ABC, we have

$$\angle A = 30^\circ, \qquad \tan(\angle B) = \frac{\sqrt{3}}{2}, \qquad AC = 2\sqrt{3}.$$

Determine the length of AB.

A. 5 B. $\sqrt{3}$ C. $2+\sqrt{3}$ D. $\frac{7}{2}$ E. $2\sqrt{7}$

Solution: A. 5



We have

$$CD = AC\sin(30^\circ) = 2\sqrt{3}\cdot\frac{1}{2} = \sqrt{3}.$$

Now

$$BD = \frac{CD}{\tan(B)} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2.$$

And

$$AD = \sqrt{(2\sqrt{3})^2 - (\sqrt{3})^2} = \sqrt{9} = 3.$$

So

$$AB = AD + DB = 3 + 2 = 5.$$

10. Suppose that 55 students were polled to determine the types of music they had stored on their phones; country, rock, or hip-hop.

Of the three genres, hip-hop was most popular, stored by 30 students; followed by 25 students who stored rock, and 10 students who stored country.

Only 4 students had stored songs from all three genres.

It was also found that 15 students who stored both rock and hip-hop had stored no country.

8 students who stored both rock and country had stored no hip-hop.

4 students who stored both hip-hop and country had stored no rock.

How many of these 55 students did not store any of these three genres.

A. 3 B. 10 C. 15 D. 8 E. 13

Solution: E. 13

Let U be the set of all 55 students, C be the set of students who have downloaded country music, R the set of students who have downloaded rock music, and H be the set of students who downloaded hip-hop.

Then $C \cup R \cup H$ is the set of all students who downloaded at least one of the three genres. Using inclusion and exclusion we have

$$|C \cup R \cup H| = |C| + |R| + |H| - |C \cap R| - |C \cap H| - |R \cap H| + |C \cap R \cap H|$$

$$= 30 + 25 + 10 - 15 - 8 - 4 + 4 = 42.$$

Then the number of students with no such music is

$$|U| - |C \cup R \cup H| = 55 - 42 = 13.$$

11. Find the coefficient of a^4b^6 in the expansion of $(a + 2b)^{10}$.

A. 210 B. 13, 440 C. 151, 200 D. 5, 040 E. 420

Solution: B. 13, 440

Using the Binomial Theorem we have, $(a+2b)^{10} = \sum_{k=0}^{10} \binom{10}{k} a^{10-k}(2b)^k$. The coefficient of a^4b^6 can be found by plugging in $k = 6$, and the coefficient is $\binom{10}{6} 2^6 = (210)(64) = 13,440$.

12. ($\star$) Simplify

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

A. 10 B. 9 C. 3 D. $\frac{1}{9}$ E. $\frac{1}{3}$

Solution: B. 9

Rationalizing each term we see

$$\begin{aligned}
\frac{1}{\sqrt{1}+\sqrt{2}} \cdot \frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}} &= \sqrt{2}-\sqrt{1}, \\
\frac{1}{\sqrt{2}+\sqrt{3}} \cdot \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} &= \sqrt{3}-\sqrt{2}, \\
&\vdots \\
\frac{1}{\sqrt{99}+\sqrt{100}} \cdot \frac{\sqrt{99}-\sqrt{100}}{\sqrt{99}-\sqrt{100}} &= \sqrt{100}-\sqrt{99}.
\end{aligned}$$

So our sum becomes

$$(\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + (\sqrt{100}-\sqrt{99})$$

$$= \sqrt{100} - \sqrt{1} = 10 - 1 = 9.$$

13. The top of a coffee table is the shape of a rectangle. The longer side measures 28 inches and it's diagonal measures 35 inches. How long (in inches) is the shorter side?

A. 18 B. 20 C. 21 D. 22 E. 24

Solution: C. 21

We have a right triangle with side 28 and hypotenuse 35. Let x be the shorter side. Using the Pythagorean Theorem we have $35^2 = 28^2 + x^2$. So,

$$x^2 = 441 \Rightarrow x = \pm 21.$$

Note: We also could've noticed this is a 3-4-5 triangle multiplied by 7. Thus the shorter side is $3 \times 7 = 21$.

14. Let $n \geq 2$ be any natural number. Which of the following must be divisible by 3?

A. $n(n^2 - 1)$ B. $n^2(n+1)$ C. $n^2 + 3n + 2$ D. All of these E. None of these

Solution:

A. $n(n^2 - 1)$

For A we have $n(n^2 - 1) = n(n-1)(n+1)$. If we have 3 consecutive integers, then at least one of them must be divisible by 3. We see this is true because n can have remainder $0, 1,$ or 2 when divided by 3. If its remainder is 0, then n is divisible by 3, as is the product. If its remainder is 1 then $n - 1$ will have remainder of 0 when divided by 3. Thus the product will be divisible by 3. If its remainder is 2 then $n + 1$ will have a remainder of 0 when divided by 3. Thus the product will be divisible by 3. In every case we see the product is divisible by 3.

For B when $n = 4$, we have $4^2(4 + 1) = 16(5) = 80$, which is not divisible by 3.

For C when $n = 3$, we have $3^2 + 3(3) + 2 = 20$, which is not divisible by 3.

So we see only A holds true for all $n \geq 2$.

15. Two guests are the first to arrive at a party. A second group of guests arrives containing three more people than the first group did. A third group arrives containing three more people than the second group did. A fourth group arrives containing three more people than the third group did. This pattern continues until twenty groups have arrived. How many people have arrived at the party after all twenty groups are present?

A. 610 B. 640 C. 560 D. 230 E. 440

Solution: A. 610

The number of guests, g, is

$$\begin{aligned} g &= 2+(2+3)+(2+2(3))+(2+3(3))+(2+4(3))+(2+5(3))+\cdots(2+19(3)) \\ &= 2(20)+3(1+2+3+4+\cdots+19) \\ &= 40+3\left(\frac{19(20)}{2}\right) \\ &= 40+3(190) \\ &= 610. \end{aligned}$$

16. We can write 2016 as

$$2016 = n+(n+1)+(n+2)+\cdots+61+62+63.$$

What is n?

A. 1 B. 2 C. 3 D. 4 E. 5

Solution: A. 1

Note that if n is even then the number of terms in the sum is even, and if n is odd then the number of terms in the sum is odd.

If the number of terms is even, we can rearrange the sum as

$$\begin{aligned} &(n+63)+((n+1)+62)+((n+2)+61)+\cdots \\ &= \underbrace{(n+63)+(n+63)+\cdots+(n+63)}_{x \text{ times}} = (n+63)x. \end{aligned}$$

When $n=2$ we would have $65x=2016$, which does not have an integer solution. When $n=4$ we would have $67x=2016$, which does not have an integer solution.

If the number of terms is odd, we can rearrange the sum as

$$\begin{aligned} &(n+63)+((n+1)+62)+((n+2)+61)+\cdots+(n+63)+\frac{n+63}{2} \\ &= \underbrace{(n+63)+(n+63)+\cdots+(n+63)}_{x \text{ times}}+\frac{n+63}{2} = (n+63)x+\frac{n+63}{2}. \end{aligned}$$

When $n=1$ we have $64x+32=2016$, so $x=31$.

Thus

$$2016 = 1+2+3+\cdots+62+63.$$

Note: A triangular number counts the objects that can form an equilateral triangle. The numbers have an explicit formula of

$$T_n = \sum_{k=1}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

Since $2016 = 1+2+\cdots+63$, it is a triangular number, the 63^{rd} triangular number to be exact.

Note that since $2016 = n+(n+1)+(n+2)+\cdots+61+62+63$ works for $n=1$, it does not hold for $n=3$ or $n=5$.

17. Assume θ is an angle in the 4^{th} quadrant such that $\tan\theta = -3/7$. Find $\sin\theta$.

A. $-\frac{3\sqrt{58}}{58}$ B. $\frac{3\sqrt{58}}{58}$ C. $-\frac{7\sqrt{58}}{58}$ D. $\frac{7\sqrt{58}}{58}$ E. $-\frac{\sqrt{58}}{3}$

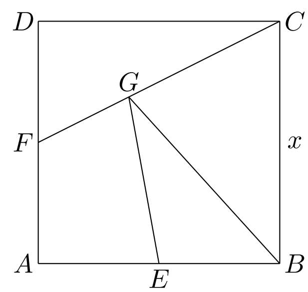
Solution: A. $-\frac{3\sqrt{58}}{58}$

We have a right right triangle in the 4^{th} quadrant with sides $-3, 7, \sqrt{58}$, as the opposite, adjacent, and hypotenuse respectively. So that

$$\sin(\theta) = -\frac{3}{\sqrt{58}} = -\frac{3\sqrt{58}}{58}.$$

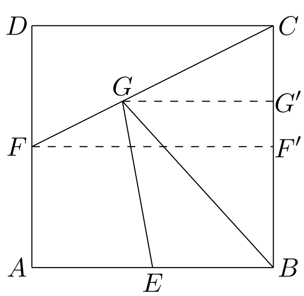
18. ($\star$)

The side length of square $ABCD$ is x. Let E and F be the midpoints of $\overline{AB}$ and $\overline{AD}$ respectively. Assume G is a point on $\overline{CF}$ and that $2\overline{CG} = 3\overline{GF}$. Determine the area of triangle BEG.



A. $\frac{3}{10}x^2$ B. $\frac{1}{6}x^2$ C. $\frac{7}{40}x^2$ D. $\frac{3}{20}x^2$ E. $\frac{2}{3}x^2$

Solution: C. $\frac{7}{40}x^2$



We have Area$(BEG) = \frac{1}{2}\overline{EB} \cdot \overline{G'B}$ and $\overline{EB} = \frac{1}{2}\overline{AB} = \frac{1}{2}x$.

We are given $2\overline{CG} = 3\overline{GF}$, so $\overline{GF} = \frac{2}{3}\overline{CG}$.

Now

$$\overline{CF} = \overline{CG} + \overline{GF} = \overline{CG} + \frac{2}{3}\overline{CG} = \frac{5}{3}\overline{CG}.$$

Since $\triangle CGG'$ is similar to $\triangle CFF'$,

$$\frac{\overline{CG'}}{\overline{CF'}} = \frac{\overline{CG}}{\overline{CF}} = \frac{\overline{CG}}{5/3\overline{CG}} = \frac{3}{5}.$$

Therefore,

$$\overline{CG'} = \frac{3}{5}\overline{CF'} = \frac{3}{5} \cdot \frac{x}{2} = \frac{3x}{10}.$$

And
$$\overline{G'B} = \overline{CB} - \overline{CG'} = x - \frac{3x}{10} = \frac{7}{10}x.$$
So
$$\text{Area}(BEG) = \frac{1}{2} \cdot \frac{1}{2}x \cdot \frac{7}{10}x = \frac{7}{40}x^2.$$

19. Consider a tournament with four teams. Each team plays every other team exactly once. In each game, the winning team earns 3 points and the losing team earns no points. If there is a tie, then both teams earn 1 point. After the tournament, the two teams with the highest numbers of points win prizes (if several teams are tied for second place, a coin flip determines which of those teams wins the prize). What is the *least* number of points a team can earn and still win a prize?

A. 2 B. 3 C. 4 D. 5 E. 6

Solution: A. 2

The least number of points is 2. There exists a scenario where 2 points is good enough to finish in (a tie for) second place, but in all possible scenarios, earning fewer than 2 points means you won't get a prize.

Suppose Team A wins every game, and teams B, C, and D (who all lost to team A) tie each other. Then Team A finishes with 9 points, and the other three teams each finish with 2 points. Thus one of teams B, C, or D must win a prize.

On the other hand, suppose Team X has fewer than 2 points. Then at best, team X tied one game and lost 2 games. Thus there are two other teams that beat team X and therefore have at least three points each. That means team X finishes 3rd or worse, and gets no prize.

20. A fair coin is flipped twice in succession. What is the probability that the second flip is a "heads," given that at least one "heads" was flipped?

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{1}{3}$ D. $\frac{1}{4}$ E. $\frac{3}{4}$

Solution: B. 2/3

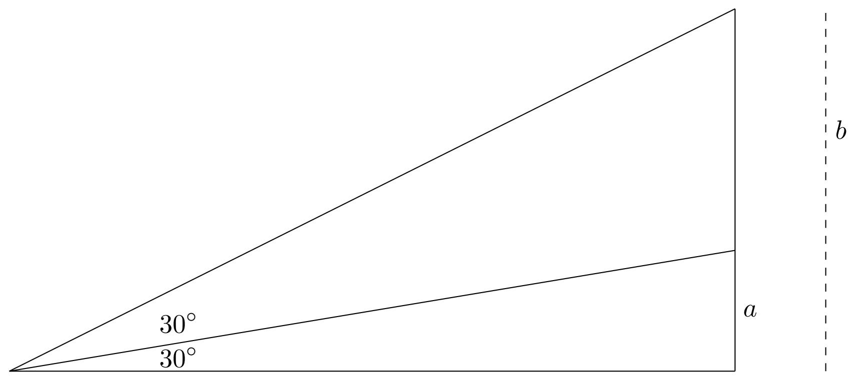
The options to get at least one "heads" are: HH, HT, or TH. We see that $\frac{2}{3}$ of the times the second flip is "heads".

We can also think of using conditional probabilities. Let A be the event that the second flip is "heads" and B be the event that at least one "heads" was flipped. Then
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/2}{3/4} = \frac{2}{3}.$$

21. A hot air balloon is rising vertically at a constant speed. An observer is standing on level ground 100 feet away from the point of launch. At one instant the observer measures the angle of elevation of the balloon as 30°. Three minutes later, the angle of elevation is 60°. What is the rate at which the balloon travels, measured in feet per minute?

A. $\dfrac{100\sqrt{3}}{3}$ B. $\dfrac{200\sqrt{3}}{3}$ C. $\dfrac{100\sqrt{3}}{9}$ D. $\dfrac{200\sqrt{3}}{9}$ E. None of these.

Solution: D. $\dfrac{200\sqrt{3}}{9}$



We have

$$\tan(30°) = \frac{a}{100} = \frac{\sqrt{3}}{3},$$

so that $a = \dfrac{100\sqrt{3}}{3}$. And

$$\tan(60°) = \frac{b}{100} = \sqrt{3}$$

so that $b = 100\sqrt{3}$.

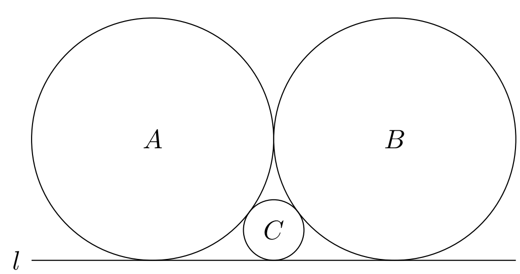
In 3 minutes, the balloon travels

$$d = a - b = 100\sqrt{3} - \frac{100\sqrt{3}}{3} = \frac{200\sqrt{3}}{3} \text{ feet.}$$

We have distance = rate × time or $d = rt$.

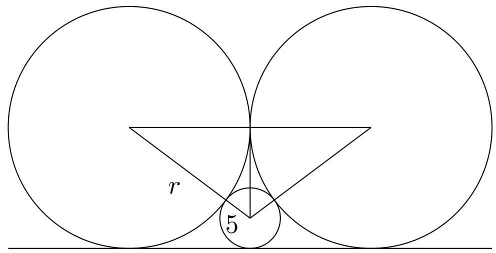
So the balloon's rate is $r = \dfrac{d}{t} = \dfrac{200\sqrt{3}}{9}$ feet per minute.

22. ($\star$) Consider the three circles below all tangent to line l and to each other. Suppose circles A and B have equal radii and the radius of C is 5cm. What is the radius of A?



A. 15 B. 20 C. 25 D. 40 E. 40

Solution: B. 20



Let r be the radius of circle A. Then we have a right triangle with sides $r+5, r, r-5$ as the hypotenuse, base, and height respectively. Using the Pythagorean Theorem, we have

$$(r+5)^2 = r^2 + (r-5)^2.$$

So

$$\begin{aligned} r^2 + 10r + 25 &= r^2 + r^2 - 10r + 25 \\ r^2 - 20r &= 0 \\ r(r-20) &= 0 \end{aligned}$$

Thus we have $r = 0$ or $r = 20$. We must have $r = 20$.

23. ($\star$) How many solutions are there to the equation $a+b+c+d = 10$, if a, b, c, and d are all integers greater than or equal to 1?

A. 4 B. 24 C. 64 D. 84 E. 124

Solution: D. 84

The number 10 is being partitioned into four nonempty parts. Thinking of 10 as ten ones, we can initially give 1 to each variable. We then need to count the number of possible ways to distribute the remaining six ones. This can be counted as the number of ways to choose, from a row of 9 empty spots, 6 spots to be ones and 3 spots to be dividing lines between the four variables. Hence there are $\binom{9}{6} = 84$ solutions.

24. Find all possible values of x for $\sin x + \sin(3x) = 0$, where $0 \leq x \leq \pi$.

A. $0, \frac{\pi}{2}, \pi$ B. $0, \pi$ C. $0, \frac{\pi}{6}$ D. $\frac{\pi}{2}$ E. $0, \frac{\pi}{6}, \pi$

Solution: A. $0, \frac{\pi}{2}, \pi$

We have

$$\begin{aligned} \sin x + \sin(3x) &= \sin(3x) + \sin x \\ &= 2\sin(2x)\cos(x) \\ &= 2\sin(x)\cos(x)\cos(x) \\ &= 2\sin(x)\cos^2(x). \end{aligned}$$

Therefore $\sin(x) + \sin(3x) = 0$ if and only if $\sin x = 0$ or $\cos x = 0$. The angles satisfied are $0, \pi/2, \pi$.

25. Compute

$${}_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5 + {}_{10}C_6 + {}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10}.$$

Note: ${}_{10}C_k$ is a combination, i.e. a way of selecting k items from a collection of 10. Other common notations include $C(10, k)$ and $\binom{10}{k}$.

A. 638 B. 2048 C. 3628800 D. 772 E. 1024

Solution: E. 1024

Method 1: Note: ${}_nC_k = {}_nC_{n-k}$. So, ${}_{10}C_0 = {}_{10}C_{10}, {}_{10}C_1 = {}_{10}C_9$, etc.
We have

$$\begin{aligned} &{}_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5 + {}_{10}C_6 + {}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10} \\ &= 2({}_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4) + {}_{10}C_5 \\ &= 2\left(\frac{10!}{0!10!} + \frac{10!}{1!9!} + \frac{10!}{2!8!} + \frac{10!}{3!7!} + \frac{10!}{4!6!}\right) + \frac{10!}{5!5!} \\ &= 2(1 + 10 + 45 + 120 + 210) + 252 \\ &= 1024. \end{aligned}$$

Method 2: Note that this is the 10^{th} row of Pascal's triangle. The n^{th} row of Pascal's triangle has a sum of 2^n. That is, $\sum_{i=0}^{10} {}_{10}C_i = 2^{10} = 1024$.

26. Let $x = \sqrt{3}$. Find the value of $|x| + |x - 2|$.
A. $\sqrt{3}$ B. $2\sqrt{3}$ C. $2\sqrt{3} + 2$ D. 2 E. None of these

Solution: D. 2

We have

$$|x - 2| = \begin{cases} -(x - 2) & x \le 2 \\ x - 2 & x > 2 \end{cases}$$

Since $\sqrt{3} < 2$, $|x-2| = -(x-2) = 2-x$. So

$$|x| + |x-2| = |\sqrt{3}| + |\sqrt{3}-2| = \sqrt{3} + 2 - \sqrt{3} = 2$$

27. Penny is repeatedly rolling a six sided die she found on Leonard and Sheldon's coffee table. She's not paying much attention but Sheldon is. After her twelfth roll he squeals and says: that's the first time that you rolled a number for the third time! Penny is amazed and wants to know which number. Sheldon tells her that the sum of the twelve numbers rolled is 47. Which number occurred three times?

A. 2 B. 3 C. 4 D. 5 E. 6

Solution: E. 6

On roll 11, no number was seen three times. This means after that roll five numbers appeared twice and one appeared once. Call the number that only appeared once x.

If x was rolled on the 12^{th} roll then the total sum would be $2(1+2+3+4+5+6) = 42$. So after the 11^{th} roll, the sum is $42 - x$.

Let y be the number they rolled on the 12^{th} roll. Then we have

$$42 - x + y = 47 \Rightarrow y - x = 5.$$

The only possibility is $y = 6$ (and $x = 1$). So she has rolled six three times.

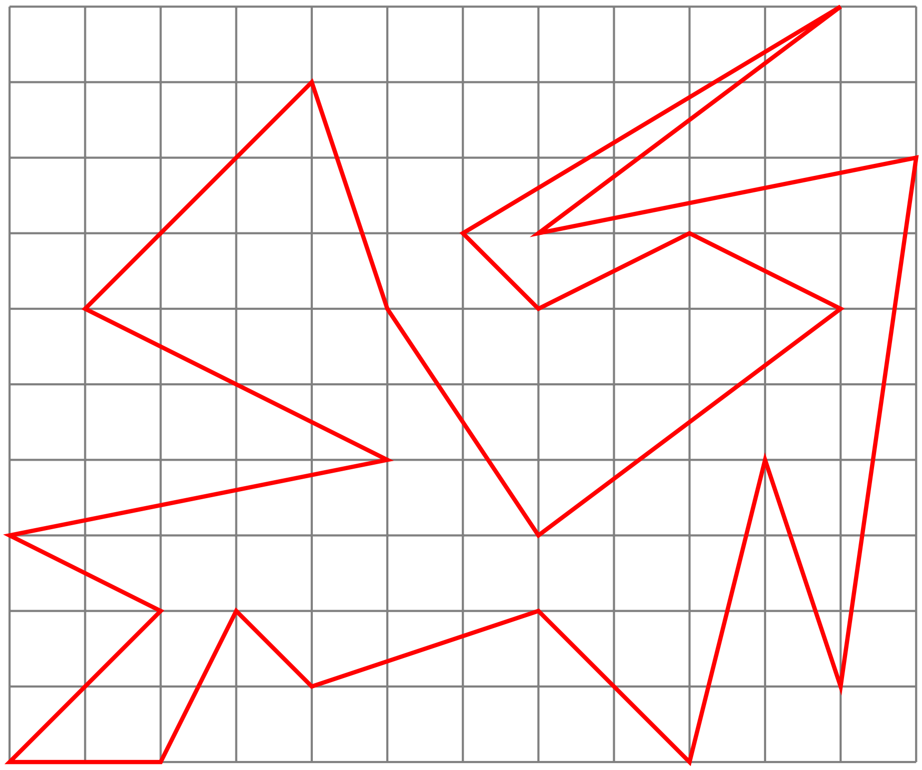
28. Let A be the point in space with coordinates $(3, 5, 2)$. Find the distance from A to the x-axis.

A. $\sqrt{21}$ B. 5 C. $\sqrt{29}$ D. $\sqrt{7}$ E. 3

Solution: C. $\sqrt{29}$

The closest point to A on the x-axis is $(3, 0, 0)$. So the distance is $\sqrt{0 + 5^2 + 2^2} = \sqrt{29}$.

29. ($\star$) Find the area of the following 22-sided polygon (assume each grid square is one square unit of area).



A. 42 B. 46 C. 48 D. 50 E. 52

Solution: C. 48

Method 1: We can partition the polygon into simpler shapes (squares and right triangles), and add the areas of the simpler shapes.

Method 2: Use Pick's Theorem, which states that if $\mathcal{P}$ is a polygon whose vertices are all integer points (i.e. points where the horizontal and vertical grid lines intersect), then its area is

$$A = \mathcal{I} + \frac{\mathcal{B}}{2},$$

where $\mathcal{I}$ is the number of integer points in the interior of $\mathcal{P}$, and $\mathcal{B}$ is the number of integer points on the boundary of $\mathcal{P}$. Here, we have $\mathcal{I} = 34$ and $\mathcal{B} = 28$, so the area of $\mathcal{P}$ is $34 + 14 = 48$.

30. An oval track measures 420 yards. Jim and Tom begin running at the same time from the same start line. Jim runs at a constant rate of 220 yards per minute while Tom runs at a constant rate of 180 yards per minute. How many laps has Tom completed when Jim passes Tom for the first time?

A. $3\frac{1}{2}$ B. $4\frac{1}{2}$ C. 5 D. $5\frac{1}{2}$ E. 6

Solution: B. $4\frac{1}{2}$

Every minute, Jim completes 40 more yards than Tom. So it will take Jim $\frac{420}{40} = 10.5$ minutes to pass Tom for the first time.

In 10.5 minutes Tom has run $10.5 \times 180 = 1890$ yards.

Therefore, he has run $\frac{1890}{420} = \frac{9}{2} = 4\frac{1}{2}$ laps.

31. A mark up on a retail item is a percentage of its cost which when added to its cost determines its selling price. How much should a retailer mark up his goods so that when he has a 50% off sale, the resulting prices will still reflect a 75% markup (on his cost)?

A. 125% B. 500% C. 250% D. 350% E. None of these

Solution: D. 350%

We have $x(.5) = 1.75$. So $x = 3.50$ or a 350% markup.

32. There are four people, Alice, Bob, Carol, and Dan. At most one person has chocolate.

- Alice says "Carol has chocolate."
- Bob says "Alice has chocolate."
- Carol says "Bob is not telling the truth."
- Dan says "I don't have chocolate."

If exactly one person is telling the truth, who has chocolate?

A. Alice B. Bob C. Carol D. Dan E. No one

Solution: D. Dan

Based on Carol's statement, we know that exactly one of Bob or Carol is telling the truth. So both Alice and Dan are not telling the truth. Since Dan is not telling the truth, his statement of "I don't have chocolate" is false. Thus Dan must have chocolate.

33. If $\log_3 A = x$, find 9^x.

A. $\sqrt{A}$ B. $A^{3/2}$ C. A D. $2A$ E. A^2

Solution: E. A^2

If $\log_3 A = x$, then $3^x = A$. So $(3^x)^2 = 9^x = A^2$.

34. If $\tan\theta = 2\sin(2\theta)$, where $0 \le \theta \le \pi$, what is θ?

A. $\theta = \frac{\pi}{6}$ B. $\theta = \frac{\pi}{3}$ C. $\theta = \frac{\pi}{4}$ D. $\theta = \frac{5\pi}{6}$ E. $\theta = \frac{3\pi}{4}$

Solution: B. $\theta = \dfrac{\pi}{3}$

If $\tan\theta = 2\sin(2\theta)$, then $\dfrac{\sin\theta}{\cos\theta} = 4\sin\theta\cos\theta$, and so $\dfrac{1}{\cos\theta} = 4\cos\theta$, and $\cos^2\theta = \dfrac{1}{4}$. Thus $\cos\theta = \dfrac{1}{2}$ or $\cos\theta = -\dfrac{1}{2}$. Hence, $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}, \ldots$

35. The lengths of three sides of a right triangle form a geometric progression. If the shortest side has length 1, then what is the length of the hypotenuse?

A. $\sqrt{5} - 1$ B. $\dfrac{1+\sqrt{5}}{2}$ C. $\dfrac{1+2\sqrt{5}}{2}$ D. $2\sqrt{5} - 1$ E. None of these.

Solution: B. $\dfrac{1+\sqrt{5}}{2}$

Let a denote the length of the longer side. Since the lengths are a geometric progression, the length of the hypotenuse is a^2. By the Pythagorean Theorem we have $1 + a^2 = a^4$. Letting $x = a^2$, we have

$$x^2 - x - 1 = 0.$$

Solving using the quadratic formula, we have

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

The positive real root is $x = \dfrac{1+\sqrt{5}}{2}$.