2016 King's College Math Competition

King's College welcomes you to this year's mathematics competition and to our campus. We wish you success in this competition and in your future studies.

Instructions

This is a 90-minute, 35-problem multiple-choice exam with no calculators allowed. There are five possible responses to each question. You may mark the test booklet and on the paper provided to you. If you need more paper or an extra pencil, let one of the monitors know. When you are sure of an answer, circle the answer on the exam. Then carefully write your answer on the score sheet with a **capital** letter. If your answer is unreadable, then the question will be scored as incorrect. The examination will be scored on the basis of 7 points for each correct answer, 2 points for each omitted answer, and 0 points for each incorrect response. Note that wild guessing is likely to lower your score.

Pre-selected problems will be used as tie-breakers for individual awards. These problems designated by ($\star$). The problems are numbered: 12, 18, 22, 23, 29

Review and check your score sheet carefully. Your name and school name should be clearly written on your score sheet.

When you complete your exam, bring your pencil, scratch paper, and answer sheet to the scoring table. You may keep your copy of the exam. Your teacher will be given a copy of the solutions to the exam problems.

Do not open your test until instructed to do so!

Good luck!

1. Three students were asked to add two positive numbers. The first student multiplied the two numbers and got 165. The second student subtracted the two numbers and got 28. If the third student was the only one who got the correct answer, what did her answer have to be?

   A. 33 B. 38 C. 5 D. 28 E. 17

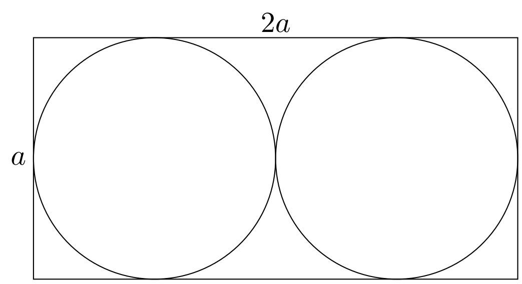
2. A girl who is 4 ft. tall is standing next to a telephone pole. At 2 PM her shadow is 7 ft. long and the telephone pole's shadow is 31.5 ft. long. How tall (in feet) is the telephone pole?

   A. 126 B. 55.125 C. 36 D. 28 E. 18

3. Consider four integers $a, b, a-b$, and $a+b$ that are all prime numbers. Which of the following is true about the sum of all four integers?

   A. divisible by 2 B. divisible by 3 C. divisible by 5 D. divisible by 7 E. none of these

4. Consider two circles of equal radius tangent to each other inscribed in a rectangle with width a and length $2a$ as shown below.



   Let X denote the total area of the two circles and Y denote the area of the rectangle that is **not** contained inside either circle. Find $\frac{X}{Y}$.

   A. $\frac{\frac{\pi}{2}}{4-\pi}$ B. $\frac{\frac{3\pi}{4}}{4-\pi}$ C. $\frac{\pi}{4-\pi}$ D. $\frac{\frac{3\pi}{2}}{4-\pi}$ E. None of these.

5. We can express 2016 in the form $2^x(2^n-1)$, for some n. What is x?

   A. $2n-1$ B. $n-1$ C. $n-2$ D. $2n$ E. n^2

6. Consider

   $$WELOVEMATHEMATICSWELOVEMATHEMATICS\ldots$$

   If this pattern continues indefinitely, what will be the 203^{rd} letter in the pattern?

   A. S B. W C. T D. C E. I

7. The wheels on one bike have a radius of 1/2 ft. The wheels on a larger bike have a radius of 1 ft. If bikers using these two bikes both ride for one mile (5280 ft), how many more complete rotations will the smaller bike's wheels make than the wheels on the larger bike?

   A. 840 B. 980 C. 1680 D. 2640 E. 16579

8. Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We call a set of three numbers *acceptable* if it has at most one odd number. For example $\{1, 2, 6\}$ and $\{2, 4, 6\}$ are both acceptable. If we randomly choose three numbers from set A, what is the probability the set will be acceptable?

   A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $\frac{1}{21}$ D. $\frac{5}{14}$ E. $\frac{17}{42}$

9. In triangle ABC, we have

$$\angle A = 30^\circ, \qquad \tan(\angle B) = \frac{\sqrt{3}}{2}, \qquad AC = 2\sqrt{3}.$$

   Determine the length of AB.

   A. 5 B. $\sqrt{3}$ C. $2+\sqrt{3}$ D. $\frac{7}{2}$ E. $2\sqrt{7}$

10. Suppose that 55 students were polled to determine the types of music they had stored on their phones; country, rock, or hip-hop.

    Of the three genres, hip-hop was most popular, stored by 30 students; followed by 25 students who stored rock, and 10 students who stored country.

    Only 4 students had stored songs from all three genres.

    It was also found that 15 students who stored both rock and hip-hop had stored no country.

    8 students who stored both rock and country had stored no hip-hop.

    4 students who stored both hip-hop and country had stored no rock.

    How many of these 55 students did not store any of these three genres.

    A. 3 B. 10 C. 15 D. 8 E. 13

11. Find the coefficient of a^4b^6 in the expansion of $(a+2b)^{10}$.

    A. 210 B. 13,440 C. 151,200 D. 5,040 E. 420

12. ($\star$) Simplify

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

    A. 10 B. 9 C. 3 D. $\frac{1}{9}$ E. $\frac{1}{3}$

13. The top of a coffee table is the shape of a rectangle. The longer side measures 28 inches and it's diagonal measures 35 inches. How long (in inches) is the shorter side?

    A. 18 B. 20 C. 21 D. 22 E. 24

14. Let $n \geq 2$ be any natural number. Which of the following must be divisible by 3?

    A. $n(n^2-1)$ B. $n^2(n+1)$ C. n^2+3n+2 D. All of these E. None of these

15. Two guests are the first to arrive at a party. A second group of guests arrives containing three more people than the first group did. A third group arrives containing three more people than the second group did. A fourth group arrives containing three more people than the third group did. This pattern continues until twenty groups have arrived. How many people have arrived at the party after all twenty groups are present?

A. 610 B. 640 C. 560 D. 230 E. 440

16. We can write 2016 as

$$2016 = n + (n+1) + (n+2) + \cdots + 61 + 62 + 63.$$

What is n?

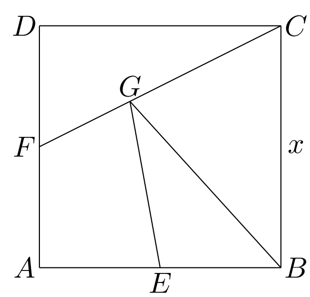
A. 1 B. 2 C. 3 D. 4 E. 5

17. Assume θ is an angle in the 4^{th} quadrant such that $\tan\theta = -3/7$. Find $\sin\theta$.

A. $-\dfrac{3\sqrt{58}}{58}$ B. $\dfrac{3\sqrt{58}}{58}$ C. $-\dfrac{7\sqrt{58}}{58}$ D. $\dfrac{7\sqrt{58}}{58}$ E. $-\dfrac{\sqrt{58}}{3}$

18. ($\star$)

The side length of square $ABCD$ is x. Let E and F be the midpoints of $\overline{AB}$ and $\overline{AD}$ respectively. Assume G is a point on $\overline{CF}$ and that $2\overline{CG} = 3\overline{GF}$. Determine the area of triangle BEG.



A. $\dfrac{3}{10}x^2$ B. $\dfrac{1}{6}x^2$ C. $\dfrac{7}{40}x^2$ D. $\dfrac{3}{20}x^2$ E. $\dfrac{2}{3}x^2$

19. Consider a tournament with four teams. Each team plays every other team exactly once. In each game, the winning team earns 3 points and the losing team earns no points. If there is a tie, then both teams earn 1 point. After the tournament, the two teams with the highest numbers of points win prizes (if several teams are tied for second place, a coin flip determines which of those teams wins the prize). What is the *least* number of points a team can earn and still win a prize?

A. 2 B. 3 C. 4 D. 5 E. 6

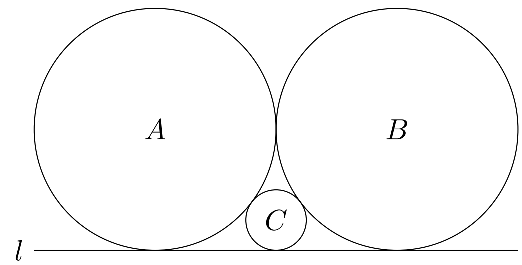
20. A fair coin is flipped twice in succession. What is the probability that the second flip is a "heads," given that at least one "heads" was flipped?

A. $\dfrac{1}{2}$ B. $\dfrac{2}{3}$ C. $\dfrac{1}{3}$ D. $\dfrac{1}{4}$ E. $\dfrac{3}{4}$

21. A hot air balloon is rising vertically at a constant speed. An observer is standing on level ground 100 feet away from the point of launch. At one instant the observer measures the angle of elevation of the balloon as 30°. Three minutes later, the angle of elevation is 60°. What is the rate at which the balloon travels, measured in feet per minute?

    A. $\frac{100\sqrt{3}}{3}$ B. $\frac{200\sqrt{3}}{3}$ C. $\frac{100\sqrt{3}}{9}$ D. $\frac{200\sqrt{3}}{9}$ E. None of these.

22. ($\star$) Consider the three circles below all tangent to line l and to each other. Suppose circles A and B have equal radii and the radius of C is 5cm. What is the radius of A?



    A. 15 B. 20 C. 25 D. 40 E. 40

23. ($\star$) How many solutions are there to the equation $a + b + c + d = 10$, if a, b, c, and d are all integers greater than or equal to 1?

    A. 4 B. 24 C. 64 D. 84 E. 124

24. Find all possible values of x for $\sin x + \sin(3x) = 0$, where $0 \leq x \leq \pi$.

    A. $0, \frac{\pi}{2}, \pi$ B. $0, \pi$ C. $0, \frac{\pi}{6}$ D. $\frac{\pi}{2}$ E. $0, \frac{\pi}{6}, \pi$

25. Compute

$$_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5 + {}_{10}C_6 + {}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10}.$$

    Note: ${}_{10}C_k$ is a combination, i.e. a way of selecting k items from a collection of 10. Other common notations include $C(10, k)$ and $\binom{10}{k}$.

    A. 638 B. 2048 C. 3628800 D. 772 E. 1024

26. Let $x = \sqrt{3}$. Find the value of $|x| + |x - 2|$.

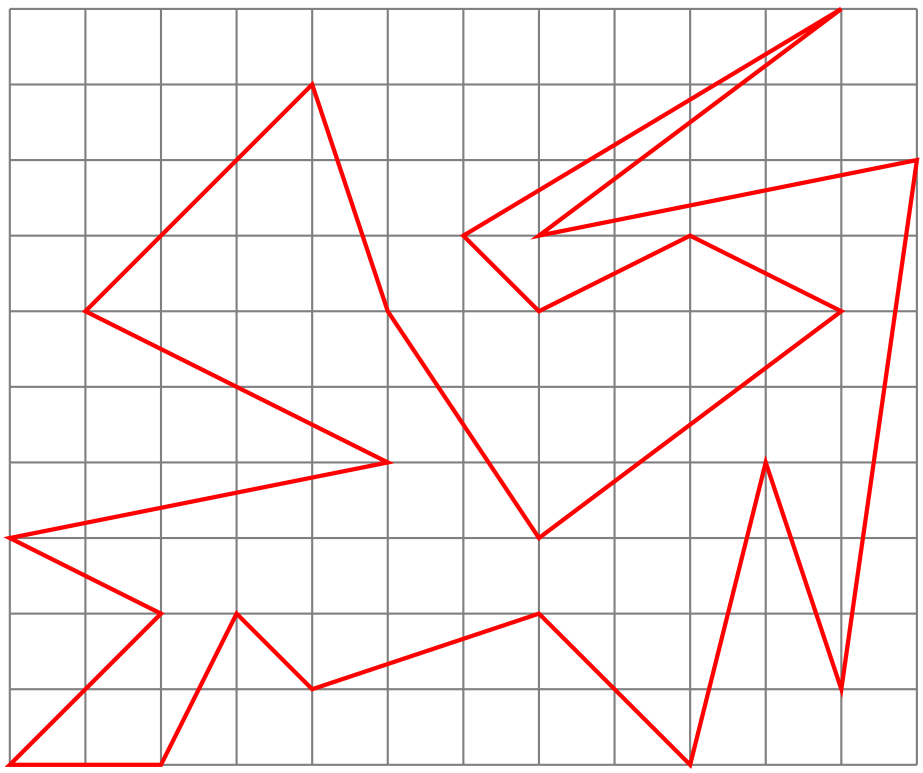
    A. $\sqrt{3}$ B. $2\sqrt{3}$ C. $2\sqrt{3} + 2$ D. 2 E. None of these

27. Penny is repeatedly rolling a six sided die she found on Leonard and Sheldon's coffee table. She's not paying much attention but Sheldon is. After her twelfth roll he squeals and says: that's the first time that you rolled a number for the third time! Penny is amazed and wants to know which number. Sheldon tells her that the sum of the twelve numbers rolled is 47. Which number occurred three times?

    A. 2 B. 3 C. 4 D. 5 E. 6

28. Let A be the point in space with coordinates $(3, 5, 2)$. Find the distance from A to the x-axis.
    A. $\sqrt{21}$ B. 5 C. $\sqrt{29}$ D. $\sqrt{7}$ E. 3

29. ($\star$) Find the area of the following 22-sided polygon (assume each grid square is one square unit of area).



A. 42 B. 46 C. 48 D. 50 E. 52

30. An oval track measures 420 yards. Jim and Tom begin running at the same time from the same start line. Jim runs at a constant rate of 220 yards per minute while Tom runs at a constant rate of 180 yards per minute. How many laps has Tom completed when Jim passes Tom for the first time?
    A. $3\frac{1}{2}$ B. $4\frac{1}{2}$ C. 5 D. $5\frac{1}{2}$ E. 6

31. A mark up on a retail item is a percentage of its cost which when added to its cost determines its selling price. How much should a retailer mark up his goods so that when he has a 50% off sale, the resulting prices will still reflect a 75% markup (on his cost)?
    A. 125% B. 500% C. 250% D. 350% E. None of these

32. There are four people, Alice, Bob, Carol, and Dan. At most one person has chocolate.

    - Alice says "Carol has chocolate."
    - Bob says "Alice has chocolate."
    - Carol says "Bob is not telling the truth."
    - Dan says "I don't have chocolate."

    If exactly one person is telling the truth, who has chocolate?

    A. Alice B. Bob C. Carol D. Dan E. No one

33. If $\log_3 A = x$, find 9^x.

    A. $\sqrt{A}$ B. $A^{3/2}$ C. A D. $2A$ E. A^2

34. If $\tan\theta = 2\sin(2\theta)$, where $0 \le \theta \le \pi$, what is θ?

    A. $\theta = \frac{\pi}{6}$ B. $\theta = \frac{\pi}{3}$ C. $\theta = \frac{\pi}{4}$ D. $\theta = \frac{5\pi}{6}$ E. $\theta = \frac{3\pi}{4}$

35. The lengths of three sides of a right triangle form a geometric progression. If the shortest side has length 1, then what is the length of the hypotenuse?

    A. $\sqrt{5} - 1$ B. $\frac{1+\sqrt{5}}{2}$ C. $\frac{1+2\sqrt{5}}{2}$ D. $2\sqrt{5} - 1$ E. None of these.